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SECOND PUBLIC EXAMINATION

Honour School of Physics Part C: 4 Year Course

Honour School of Physics and Philosophy Part C

C6: THEORETICAL PHYSICS

TRINITY TERM 2011

Friday, 17 June, 9.30 am – 12.30 pm

Answer **four** *questions.*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

1. A small particle suspended in fluid is free to rotate about one axis. Its rotation angle $\theta(t)$ changes with time t at a rate $\omega(t)$. The time evolution of this rate is described by the Langevin equation

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = -\gamma\omega(t) + \eta(t)\,,$$

where $\eta(t)$ fluctuates randomly in time with average values $\langle\eta(t)\rangle = 0$ and $\langle\eta(t_1)\eta(t_2)\rangle = \Gamma\delta(t_1 - t_2)$. Explain the physical origin of the two terms on the right-hand side of this equation and discuss the circumstances under which the assumed form for $\langle\eta(t_1)\eta(t_2)\rangle$ is appropriate. [6]

The particle is stationary with rotation angle zero at $t = 0$. Derive an expression for the rotation rate at subsequent times in terms of $\eta(t)$ and show that

$$\langle\omega(t_1)\omega(t_2)\rangle = \frac{\Gamma}{2\gamma}\left[\mathrm{e}^{-\gamma|t_1-t_2|} - \mathrm{e}^{-\gamma(t_1+t_2)}\right]$$

for $t_1, t_2 \geq 0$. Describe how this result enables one to relate the values of Γ and γ to the temperature and the moment of inertia of the particle. [10]

Using the definition

$$\theta(t) = \int_0^t \mathrm{d}t'\omega(t')$$

calculate $\langle\theta(t)\rangle$ and show that

$$\langle\theta^2(t)\rangle = \frac{\Gamma}{\gamma^2}\left[t + \frac{1}{2\gamma}(1 - \mathrm{e}^{-2\gamma t}) - \frac{2}{\gamma}(1 - \mathrm{e}^{-\gamma t})\right]\,.$$

Discuss the behaviour of $\langle\theta^2(t)\rangle$ for $\gamma t \ll 1$ and $\gamma t \gg 1$. On what timescale does the orientation of the molecule become roughly uniformly distributed? [9]

2. A model for a one-dimensional system in classical statistical mechanics has variables n_i located at the sites of a lattice, which are labelled by integers i. The variables have S possible values, so that $n_i = 0, 1 \ldots S-1$. The lattice has N sites and periodic boundary conditions, so that $i + N \equiv i$. The system has an interaction energy $\mathcal{E}(n_i, n_{i+1})$ between neighbouring sites and the total energy of a configuration is

$$H = \sum_{i=0}^{N-1} \mathcal{E}(n_i, n_{i+1}) \, .$$

A physical observable is represented by the function $c(n_i)$. Discuss how the transfer matrix approach may be used to calculate the free energy per site and the thermal averages $\langle c(n_l) \rangle$ and $\langle c(n_l) c(n_{l+m}) \rangle$ for large N. [12]

For a lattice gas $S = 2$ and

$$\mathcal{E}(n_i, n_{i+1}) = -J n_i n_{i+1} - \frac{\mu}{2}(n_i + n_{i+1}) \, .$$

Explain the physical significance of the constants J and μ. [4]

Show that

$$\langle n_i \rangle = \frac{1}{1 + \mathrm{e}^{-2\theta}}$$

where

$$\sinh \theta = \frac{1}{2} \left(\mathrm{e}^{\beta(J+\mu/2)} - \mathrm{e}^{-\beta\mu/2} \right) \, .$$

Sketch the dependence of $\langle n_i \rangle$ on μ at fixed inverse temperature β, (i) for $0 \leq \beta J \ll 1$, and (ii) for $\beta J \gg 1$. How would you expect these graphs to differ for a similar model in higher dimensions? [9]

3. The group $SU(3)$ is defined as the set of complex 3×3 matrices U satisfying $U^\dagger U = \mathbf{1}$ and $\det(U) = 1$.

(a) Show that the Lie algebra of $SU(3)$ consists of traceless, hermitian 3×3 matrices. What is the real dimension of this Lie algebra? [6]

(b) A triplet $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$ of complex scalar fields transforms under the fundamental $SU(3)$ representation. Show that the Lagrangian density

$$\mathcal{L} = \partial_\mu \boldsymbol{\phi}^\dagger \partial^\mu \boldsymbol{\phi} - V(\boldsymbol{\phi}) \ , \quad V(\boldsymbol{\phi}) = m^2 \boldsymbol{\phi}^\dagger \boldsymbol{\phi} + \frac{\lambda}{4} (\boldsymbol{\phi}^\dagger \boldsymbol{\phi})^2 \ ,$$

where m and λ are constants, is $SU(3)$ invariant. [4]

(c) Assume that $m^2 < 0$ and analyse spontaneous symmetry breaking for the theory given in (b). Choose a convenient vacuum state and find the unbroken symmetry group. How many Goldstone bosons do you expect from Goldstone's theorem in this case? [7]

(d) Verify Goldstone's theorem by explicitly working out the masses around the vacuum chosen in (c). [8]

4. The Lagrangian density for a classical vector field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ .$$

(a) Derive the equation of motion for the vector field A_μ from this Lagrangian density. Show that in Lorentz gauge, defined by $\partial_\mu A^\mu = 0$, it takes the form $\Box A_\mu = 0$. [4]

(b) Consider the plane wave ansatz $A_\mu(x) = c_\mu \sin(kx)$ where c_μ and k_μ are constant four-vectors. Under which conditions is this ansatz in Lorentz gauge and a solution to $\Box A_\mu = 0$? [6]

(c) Apply a gauge transformation with parameter $\Lambda = -\lambda \cos(kx)$, where λ is a constant, to the plane wave solution in (b) and show that it results in $c_\mu \to c_\mu + \lambda k_\mu$. What is the interpretation of this gauge transformation? For a given four-momentum k, state how many physical degrees of freedom are contained in c_μ and justify your answer. [7]

(d) The energy momentum tensor of a vector field is defined by $T_{\mu\nu} = -F_\mu{}^\rho F_{\nu\rho} + \frac{1}{4}\eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$. Compute this energy momentum tensor for the plane wave solutions obtained in (b) and verify that it satisfies $k^\mu T_{\mu\nu} = 0$. What is the interpretation of this last relation? [8]

5. The oscillator expansion for a vector field A_μ in Lorentz gauge is given by

$$A_\mu(x) = \int \mathrm{d}^3\tilde{k} \left(a_\mu(k)\mathrm{e}^{-\mathrm{i}kx} + a_\mu^\dagger(k)\mathrm{e}^{\mathrm{i}kx} \right) .$$

(a) Show that the annihilation operators $a_\mu(q)$ can be written as

$$a_\mu(q) = \int \mathrm{d}^3x \, \mathrm{e}^{\mathrm{i}qx} \left(w_{\mathbf{q}} A_\mu(x) + \mathrm{i}\partial_0 A_\mu(x) \right) ,$$

where $q = (w_{\mathbf{q}}, \mathbf{q})$ and $w_{\mathbf{q}} = |\mathbf{q}|$. [Hint: Start by performing a three-dimensional spatial Fourier transformation on $A_\mu(x)$ and $\partial_0 A_\mu(x)$.] [9]

(b) Write the annihilation operators as $a_\mu(k) = \sum_{\alpha=0}^{3} \epsilon_\mu^{(\alpha)}(k) a^{(\alpha)}(k)$ with the standard polarisation vectors $\epsilon_\mu^{(\alpha)}(k)$ which satisfy $\epsilon_\mu^{(\alpha)}(k)\epsilon^{(\beta)\mu}(k) = \eta^{\alpha\beta}$. Briefly explain the purpose of this decomposition and the physical meaning of the different polarisations. Use the result in (a) to express $a^{(\alpha)\dagger}(k)$ for $\alpha = 1, 2$ in terms of the field operator $A_\mu(x)$ and its time derivative. [4]

(c) Consider an S-matrix element ${}_{\mathrm{out}}\langle f|i, (k, \alpha)\rangle_{\mathrm{in}}$ with a vector particle with momentum k and polarisation $\alpha = 1, 2$ in the "in" state and i and f denoting an arbitrary number of other particles in the "in" and "out" state, respectively. Briefly explain (verbally) the main steps in the LSZ reduction necessary to remove the vector particle from the "in" state. [4]

(d) Carry out the LSZ reduction discussed in (c) and thereby show that

$${}_{\mathrm{out}}\langle f|i, (k, \alpha)\rangle_{\mathrm{in}} = -\mathrm{i} \int \mathrm{d}^4x \, \mathrm{e}^{-\mathrm{i}kx} \epsilon_\mu^{(\alpha)}(k) \, \Box_x \, {}_{\mathrm{out}}\langle f|A^\mu(x)|i\rangle_{\mathrm{in}} .$$ [8]

6. A massless real scalar field ϕ with a cubic interaction is described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + \frac{g}{6}\phi^3 ,$$

where g is a constant.

(a) By calculating the three-point Green function $\mathcal{G}^{(3)}(z_1, z_2, z_3)$ at order g and subsequently transforming to momentum space show that the amputated three-point function is given by

$$\tilde{\mathcal{G}}^{(3)}_{\text{amp}}(p_1, p_2, p_3) = \mathrm{i}g \ .$$ [7]

(b) Consider the scattering of two incoming particles with momenta k_1 and k_2 into two outgoing particles with momenta q_1 and q_2. By applying Feynman rules, show that at order g^2 the matrix element for this process is

$$\mathcal{M} = -g^2 \left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right] ,$$

where $s = (k_1 + k_2)^2$, $t = (k_1 - q_1)^2$ and $u = (k_1 - q_2)^2$ are the Mandelstam variables. Draw the three Feynman diagrams which correspond to the terms in $\mathcal{M}$. [8]

(c) Write the matrix element in (b) in terms of the total centre of mass energy E and the scattering angle θ in the centre of mass frame. [6]

(d) What is the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ for the above process? [4]

7. A complete, orthonormal set of wavefunctions $\{\varphi_\ell(\mathbf{r})\}$ labelled by $\ell = 1, 2\ldots$ forms a single-particle basis for a system of identical, spin-polarised fermions with coordinate $\mathbf{r}$. The operators $c_\ell^\dagger$ create fermions in these orbitals. The vacuum state is denoted by $|0\rangle$ and the state $|N\rangle$ is defined by

$$|N\rangle = c_N^\dagger c_{N-1}^\dagger \ldots c_1^\dagger |0\rangle \, .$$

Explain how this state can be represented as a normalised coordinate space wavefunction $\Psi(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_N)$ using a Slater determinant. [4]

The fermion annihilation operator at the point $\mathbf{r}$ may be written as $c(\mathbf{r}) = \sum_\ell \varphi_\ell(\mathbf{r}) c_\ell$. The density operator is $\rho(\mathbf{r}) = c^\dagger(\mathbf{r})c(\mathbf{r})$. Show that

$$\langle N|\rho(\mathbf{r})|N\rangle = \sum_{n=1}^{N} |\varphi_n(\mathbf{r})|^2$$

and that you obtain the same result by evaluating

$$N \times \int \mathrm{d}\mathbf{r}_2 \ldots \mathrm{d}\mathbf{r}_N |\Psi(\mathbf{r}, \mathbf{r}_2, \ldots \mathbf{r}_N)|^2 \, .$$

[8]

Prove that

$$\langle N|c_m^\dagger c_p^\dagger c_q c_n|N\rangle = \begin{cases} \delta_{mn}\delta_{pq} - \delta_{mq}\delta_{pn} & \text{if} \quad m, p \leq N \\ 0 & \text{otherwise} \end{cases}$$

and show that

$$\langle N|\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)|N\rangle = \delta(\mathbf{r}_1 - \mathbf{r}_2)\langle N|\rho(\mathbf{r}_1)|N\rangle + \frac{1}{2}\sum_{a,b=1}^{N} |\varphi_a(\mathbf{r}_1)\varphi_b(\mathbf{r}_2) - \varphi_b(\mathbf{r}_1)\varphi_a(\mathbf{r}_2)|^2 \, .$$

How would these last two results differ if the particles were bosons? [13]

8. The scalar φ denotes the magnetisation of a uniaxial ferromagnet. In Landau theory the free energy $F(\varphi)$ of the ferromagnet in a field h is taken to have the form

$$F(\varphi) = -h\varphi + \frac{a(T)}{2}\varphi^2 + \frac{u}{2n}\varphi^{2n} \, ,$$

where $a(T)$ is a function of temperature T and u is positive.

For the standard case of $n = 2$, outline the symmetry arguments leading to this form. Assume that $a(T) = \alpha(T - T_\mathrm{c})$, where α is a constant, and that u is independent of temperature. Show that Landau theory describes a phase transition at the temperature T_c and obtain the equilibrium values φ_0 of the magnetisation at zero field in both phases. What is the value of the order parameter exponent β within this calculation? [10]

In special circumstances a phase transition may be described by this theory, but with an integer value $n \geq 3$. How does β depend on the value of n? [4]

Find expressions at zero field for the susceptibility $\chi = \partial\varphi_0/\partial h|_{h=0}$ and the heat capacity in both phases. [8]

Comment on your results in the limiting case of large n. [3]