2752

SECOND PUBLIC EXAMINATION

Honour School of Physics Part C: 4 Year Course

Honour School of Physics and Philosophy Part C

C2: LASER SCIENCE AND QUANTUM INFORMATION PROCESSING

TRINITY TERM 2011

Friday, 24 June, 9.30 am – 12.30 pm

Answer **four** *questions.*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

1. What is meant by the term *mode-locking*? Explain briefly how mode-locking may be used to generate short laser pulses, and describe one method by which it can be achieved. [7]

A mode-locked laser produces a train of laser pulses, separated by the time T_{c} for a mode-locked pulse to go round the laser cavity once. Give a simple explanation for why the electric field of the mode-locked output must comprise a superposition of waves with angular frequencies which differ by $\Delta\omega = 2\pi/T_{\mathrm{c}}$. The electric field of the output pulse train may be written in the form

$$E(z,t) = \sum_p a_p \exp\left[\mathrm{i}\left(k_p z - \omega_p t + \phi_p\right)\right] \ ,$$

where p is an integer, and a_p, k_p, ω_p, ϕ_p are respectively the amplitude, angular wavenumber, angular frequency, and phase of each frequency component. Assuming that $\omega_p = \omega_{\mathrm{ce}} + p\Delta\omega$ and $\phi_p = \phi_0 + p\Delta\phi$, where ω_{ce}, ϕ_0 and $\Delta\phi$ are constants, show that the field amplitude in the plane $z = 0$ may be written as

$$E(0,t') = \exp\left\{\mathrm{i}\left[\phi_0 - \frac{\Delta\phi}{2\pi}\omega_{\mathrm{ce}}T_{\mathrm{c}}\right]\right\} \exp\left[-\mathrm{i}\left(\omega_{\mathrm{ce}} + P_0\Delta\omega\right)t'\right] \sum_q a_p \exp\left(-\mathrm{i}2\pi q\frac{t'}{T_{\mathrm{c}}}\right) \ ,$$

where $t' = t - \Delta\phi/\Delta\omega$ and $p = P_0 + q$ where P_0 and q are integers. Identify the terms in this expression associated with the envelope and carrier wave of the pulses, and sketch the form of the wave. [8]

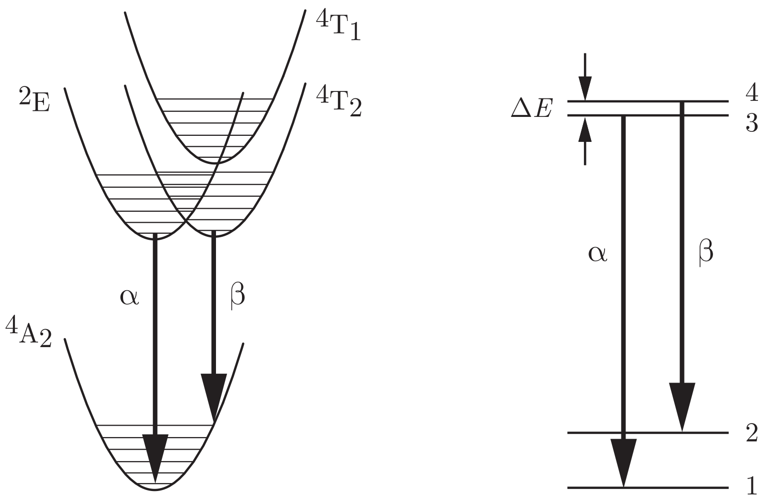
Show that for successive pulses the phase difference, ϕ_{CEO}, between the carrier wave and the envelope of the pulse changes by ϕ_{slip} where

$$|\phi_{\mathrm{slip}}| = 2\pi\frac{\omega_{\mathrm{ce}}}{\Delta\omega} \ .$$

[4]

When does the value of ϕ_{CEO} have a significant effect on the temporal properties of the mode-locked pulse? The carrier-envelope phase ϕ_{CEO} may be held constant by ensuring that the frequency difference, δ, between the frequency component $p = 2n$ and the second harmonic of the component $p = n$ is fixed at a suitable value. Explain how this method works, and state the required value of δ. [6]

2.



The figure shows a simplified configuration co-ordinate diagram of the ground state, ${}^4A_2$, and other energy levels relevant to the operation of the alexandrite ($Cr^{3+}:BeAl_2O_4$) laser. Outline how laser oscillation may be obtained on the transitions labelled α and β. For each transition discuss: whether the transition may be described as *zero-phonon* or *vibronic*; whether the laser transition operates on a three-level or four-level scheme; the spectral properties of the laser radiation; and the prospects for continuous–wave oscillation. [9]

Also shown is an energy level diagram of a simplified model of the laser transitions. Show that the pump energy which must be absorbed by the gain medium to create a population density N_3 in level 3, or N_4 in level 4, is

$$W = V_{\mathrm{g}} \frac{hc}{\lambda_{\mathrm{p}}} N_3 \left[1 + \exp\left(-\frac{\Delta E}{k_{\mathrm{B}}T}\right)\right] = V_{\mathrm{g}} \frac{hc}{\lambda_{\mathrm{p}}} N_4 \left[1 + \exp\left(+\frac{\Delta E}{k_{\mathrm{B}}T}\right)\right] ,$$

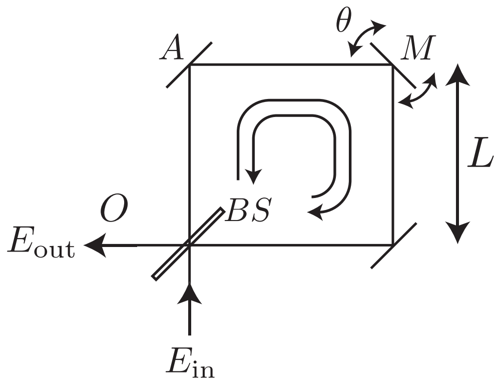
where V_{g} is the volume of the gain medium, λ_{p} is the wavelength of the pump radiation, $\Delta E = E_4 - E_3$, and E_i is the energy of level i. You may assume that the populations of levels 3 and 4 are maintained in thermal equilibrium with temperature T and that all levels have the same statistical weight. [4]

Find, in terms of the total density N_{T} of active ions, an approximate expression for the threshold pump energy, W_{th}, which must be absorbed to achieve laser oscillation on transition α. You may assume that the laser cavity only provides optical feedback on the desired laser transition. Discuss how the threshold pump energy varies with temperature, and find expressions for W_{th} in the limits of low and high temperatures. [7]

Find an expression for the minimum pump *power* required for steady-state laser oscillation on transition β. Calculate this power for the case of a cylindrical laser rod of diameter 10 mm and length 50 mm, placed in a two-mirror laser cavity with mirror reflectivities of 100% and 90%. Take the temperature of the rod to be 60°C and the wavelength of the pump radiation to be 500 nm. [5]

[For alexandrite $\Delta E = 800\,\mathrm{cm}^{-1}$, and transition β has an upper level fluorescence lifetime of 260 μs and optical gain cross-section of $7 \times 10^{-19}\,\mathrm{cm}^2$.]

3. The diagram shows a square interferometer, with side length L, designed to measure a small rotation θ of the mirror M. A wide collimated laser beam is split at the 50:50 beamsplitter (BS), so that light propagates both clockwise and anti-clockwise. The interferometer is initially aligned so that no light emerges from the output port O when $\theta = 0$.



Now the mirror at A is tilted very slightly upwards, out of the plane of the experiment, by a small angle ψ. The tilt is too small to affect the overlap of the beams at BS, but it introduces a small phase 2ϕ between the clockwise and anti-clockwise paths, so that some light emerges at the output O.

Show that $\phi \approx \pi(L/\lambda)\psi^2$, where λ is the laser wavelength, and explain why simply translating the mirror over a small distance has no effect on the output intensity. [6]

Show that, as θ is allowed to vary, the field at the output has a transverse spatial profile in the plane of the interferometer $E_{\mathrm{out}}(x)$ given by

$$E_{\mathrm{out}}(x) = \sin(\phi + k_{\perp}x)E_{\mathrm{in}}(x) \ ,$$

where $E_{\mathrm{in}}(x)$ is the transverse spatial profile of the input beam, and $k_{\perp} = (2\pi/\lambda)\sin(\theta)$. [9]

If the input beam has a Gaussian profile $E_{\mathrm{in}}(x) \propto \mathrm{e}^{-(x/\sigma)^2}$ with beam waist σ, show that for small values of x, such that $k_{\perp}x \ll 1$ and $x \ll \sigma$,

$$E_{\mathrm{out}}(x) \approx \phi E_{\mathrm{in}}(x - \langle x \rangle) \ ,$$

where $\langle x \rangle = k_{\perp}\sigma^2/2\phi$. [5]

Measuring the transverse displacement $\langle x \rangle$, with a split detector for instance, enables estimation of θ. What is the advantage of this apparatus for detecting very small rotations? What practical constraints will limit the achievable angular resolution?

[5]

4. Give a simple symmetry argument to show that a crystal must have no centre of symmetry to display a Pockels effect. [3]

For a cartesian co-ordinate system the equation of the indicatrix has the general form:

$$\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} + \frac{2yz}{n_4^2} + \frac{2xz}{n_5^2} + \frac{2xy}{n_6^2} = 1 \ .$$

For a Pockels crystal possessing tetragonal symmetry such as ADP (ammonium dihydrogen phosphate) the application of a d.c. electric field modifies the indicatrix as follows:

$$\begin{pmatrix} \Delta\left(\frac{1}{n^2}\right)_1 \\ \Delta\left(\frac{1}{n^2}\right)_2 \\ \Delta\left(\frac{1}{n^2}\right)_3 \\ \Delta\left(\frac{1}{n^2}\right)_4 \\ \Delta\left(\frac{1}{n^2}\right)_5 \\ \Delta\left(\frac{1}{n^2}\right)_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{52} & 0 \\ 0 & 0 & r_{63} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \ .$$

Use this information to show, when an electric field is applied along the z-axis, that the refractive indices along the x' and y' directions, which are given by a clockwise rotation of 45° of the original x- and y-axes about z, are given by

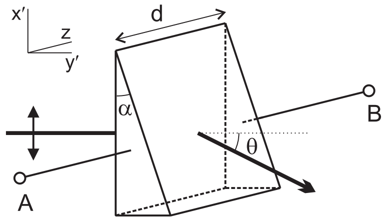
$$n_{x'}^2 = \frac{n_0^2}{1 - n_0^2 r_{63} E_z} \quad \text{and} \quad n_{y'}^2 = \frac{n_0^2}{1 + n_0^2 r_{63} E_z} \ .$$

Consider a linearly polarised laser beam of vacuum wavelength λ_0 which propagates along the z-direction with the plane of polarisation along the original y-axis of the crystal. Show that the direction of linear polarisation can be rotated by 90° by applying a voltage V_0 along z where

$$V_0 = \frac{\lambda_0}{2 n_0^3 r_{63}} \ .$$

[8]

[Question continues over page.]



A crystal of ADP is cut in the form of a right-angled triangular prism with a small apex angle α as shown in the diagram.

Given that a voltage difference of V is established between terminals A and B which is also along the z-axis of the crystal show that the deflection θ of the emergent laser beam is changed by an amount

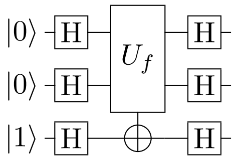
$$\Delta\theta = \left| \frac{1}{2} \alpha r_{63} n_0^3 \frac{V}{d} \right| .$$

when the direction of polarisation is along x'. How would the magnitude of the deflection change when the plane of polarisation is aligned along the z-direction? [10]

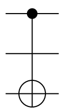
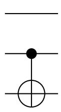
An identical second prism is inverted so that the hypotenuses of the two prisms are almost in contact with one another. Explain how the orientation of the axes and field directions can be chosen to maximise the deflection produced for a fixed voltage V.

Discuss briefly the effect diffraction has on the usefulness of such devices. [4]

5. The Deutsch–Jozsa algorithm permits the efficient identification of classical binary functions from n bits to 1 bit which are either constant or balanced, and can be implemented in the case $n = 2$ using the network below



where the last qubit is an ancilla, and the networks below act as oracle implementations of two of the six balanced functions.

$f_{0011} =$  $f_{0101} =$ 

Draw labelled networks for oracle implementations of the four remaining balanced functions, the two constant functions, and the unbalanced function f_{0001}. [7]

Find the final state of the three qubits for the two constant functions and the six balanced functions using either matrix methods (you may find it useful to factor out the ancilla qubit and any global phases) or circuit identities, and determine the probability of finding both input qubits in the final state $|0\rangle$ in each case. Repeat this calculation for the function f_{0001} and discuss whether the Deutsch–Jozsa algorithm can be useful with unbalanced functions. [13]

Describe briefly how to implement Hadamard and CNOT gates in a trapped ion quantum computer. [5]

6. Write down the matrix form of the quantum gate ϕ_z and find the effect of applying this to a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Use explicit matrix methods to show that X=HZH, and hence or otherwise find the effect of applying ϕ_x to $|\psi\rangle$. [5]

Consider an ensemble of qubits which start in the state $|\psi\rangle$ and then experience either a ϕ_z gate, an identity gate, or a ϕ_{-z} gate, chosen independently at random for each qubit in the ensemble. Show that the final state is identical to that of an ensemble of qubits which either experience a Z gate with some probability p, or are left untouched with probability $1 - p$, and find the relationship between p and ϕ. [5]

Consider the special case where $|\psi\rangle$ lies on the equator of the Bloch sphere. Calculate the purity of the final state, and find the value of ϕ which reduces the purity of the state to the minimum possible value. Use the Bloch sphere picture to explain why this occurs. [6]

Describe an error-correction network which can correct spin-flip errors using three physical qubits to encode one logical qubit. Explain why this network will also correct random ϕ_x gates, and discuss how the effectiveness varies with ϕ. How could this network be modified to correct random ϕ_z gates instead? [9]

[The purity of a density matrix ρ is defined as $\mathrm{tr}(\rho^2)$.]

7. Not all internal states of an atom are suitable for representing basis states of a qubit. State two properties of states that make a 'good' qubit. Two hyperfine states of an atom $|g\rangle$ and $|e\rangle$ with energy difference $\hbar\omega$ are used to represent a qubit. Briefly explain how coherent Rabi-oscillations can be driven between these two states. What is the effect of a resonant $\pi/2$ pulse with phase zero on the states $|g\rangle$ and $|e\rangle$? [5]

An atom is initially, at time $t = 0$, prepared in the state $|e\rangle\langle e|$, and then decays to the state $|g\rangle\langle g|$ at a rate γ. Calculate the density matrix of the atom and its entropy as a function of time and give a physical reason why this evolution cannot be unitary. What is the time evolution if the atom starts in $|g\rangle\langle g|$? The time evolution of the initial operator $|e\rangle\langle g|$ is given by

$$|e\rangle\langle g| \to \mathrm{e}^{-\mathrm{i}\omega t - \gamma t/2}|e\rangle\langle g| \; .$$

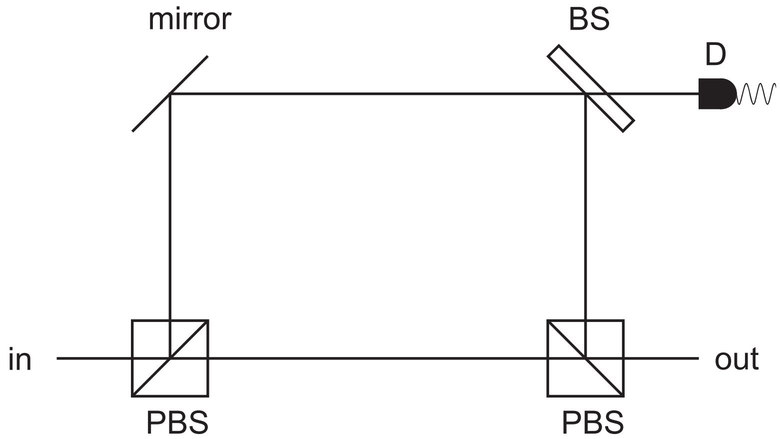
What is the evolution of the operator $|g\rangle\langle e|$?

Describe the trajectory followed by the initial density matrix $|e\rangle\langle e|$ on the Bloch sphere where the north pole corresponds to $|g\rangle\langle g|$ and the south pole to $|e\rangle\langle e|$ qualitatively as a function of t. Compare this to the trajectory of the state of an atom driven by a π pulse from $|e\rangle\langle e|$ to $|g\rangle\langle g|$. How does the entropy change along these trajectories? [10]

The atom now starts in $|g\rangle\langle g|$ and an instantaneous $\pi/2$ pulse is applied at time $t = 0$. It is then allowed to evolve for a time τ before a second instantaneous $\pi/2$ pulse is applied. The atom is measured immediately afterwards. Calculate the probability of finding the atom in state $|e\rangle$ in this measurement as a function of τ for a given ω and the cases where $\gamma = 0$ and where $\gamma \neq 0$. Briefly discuss the implications of this result for using atomic states $|g\rangle$ and $|e\rangle$ as a qubit or as the two paths of an atom interferometer. [10]

8. The computational basis states of a qubit are encoded as as horizontal $|H\rangle$ and vertical $|V\rangle$ polarizations of a photon. Which polarizations do the X and the Y basis states correspond to in this encoding? Draw schematic experimental setups using polarizing beam splitters (PBS), photo-detectors and wave-plates for measuring the qubit in each of the three bases. [5]

A two photon source produces polarization entangled photons in the state $|\alpha\rangle = \sqrt{\alpha}|VV\rangle + \sqrt{1-\alpha}|HH\rangle$ with $0 \leq \alpha \leq 1$. Calculate the joint entropy, the entropy of the reduced density matrix of each photon, and the mutual information between the photons as a function of α. Discuss the significance of the mutual information for measurements in the cases where $|\alpha\rangle$ is a product state and where it is a maximally entangled state. [8]



Alice and Bob receive one photon of $|\alpha\rangle$, respectively. Bob sends his photon through the device shown in the figure where the PBSs transmit photons in $|V\rangle$ and reflect those in state $|H\rangle$; the beam splitter (BS) has reflectivity R for both polarizations. Alice and Bob only keep those photon pairs where the detector D does not click. Work out the state of these remaining photons. How must R be chosen so that these photon pairs are maximally entangled? For which values of α is such a choice of R physically possible? If the photon source produces N pairs per second how many maximally entangled photon pairs do Alice and Bob obtain per second? [12]