2776

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 4 Year Course

B2: III. QUANTUM, ATOMIC AND MOLECULAR PHYSICS AND IV. SUB-ATOMIC PHYSICS

TRINITY TERM 2011

Friday, 24 June, 9.30 am – 12.30 pm

Answer **four** *questions,* **two** *from each section:*

Start the answer to each question in a **fresh book.**

At the end of the examination hand in your answers to Section III and Section IV in separate bundles.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section III. (Quantum, atomic and molecular physics)

1. Explain what is meant by the *central field approximation* and show how it leads to the concept of electron configurations. Show further that the application of residual electrostatic and spin-orbit interactions leads to splitting of a configuration into terms and levels. [9]

Explain what is meant by LS coupling and show how the Landé interval rule arises in the spectrum of a multi-electron atom. [6]

The low-lying electronic levels of neutral Ca (Z=20) which arise from 4s^2, 4s4p and 3d4s configurations are presented in the following table:

Configuration	Energy (cm^{-1})
4s^2	0
4s4p	15158
	15210
	15315
	23652
3d4s	20335
	20349
	20371
	20850

(i) Assign L, S, J quantum numbers to each of the levels, carefully justifying your assignments. Label each state using the conventional spectroscopic notation. [5]

(ii) Indicate the allowed optical transitions on an energy level diagram. Which transition gives rise to the strongest line in the spectrum? [5]

2. Describe the physical origin of the spin-orbit interaction, for an atom with a single valence electron. [4]

The contribution to the atomic Hamiltonian due to the spin-orbit interaction may be written as

$$\Delta\hat{H}_{\text{so}} = \frac{\hbar^2}{2m_{\text{e}}^2c^2}\,\frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}r}\,\hat{\mathbf{l}}\cdot\hat{\mathbf{s}}$$

where $\hbar\hat{\mathbf{l}}$ and $\hbar\hat{\mathbf{s}}$ are respectively the orbital and spin angular momenta of the electron, and $U(r)$ is the potential energy of the electron in the central field.

Give the angular momentum quantum numbers used to label the states arising from the spin-orbit interaction, and justify their use. [4]

For hydrogen, show that a gross structure energy level with quantum numbers n and $l \neq 0$ is split by the spin-orbit interaction into two levels separated by an energy

$$\Delta E_{\text{fs}} = \frac{\mu_0\mu_{\text{B}}^2}{2\pi}\,\frac{1}{a_0^3n^3l(l+1)}$$

where $\mu_{\text{B}} = e\hbar/2m_{\text{e}}$ is the Bohr magneton. [9]

The 4p term in potassium ($Z = 19$) has a binding energy of $-2.73\,\text{eV}$ and a fine structure splitting of $0.00716\,\text{eV}$. Estimate the binding energy and the fine structure splitting of the 5p term in this atom. [8]

[For hydrogen,

$$\left\langle\frac{1}{r^3}\right\rangle = \frac{1}{a_0^3n^3l(l+\frac{1}{2})(l+1)}$$

where a_0 is the Bohr radius.]

3. Consider the interaction of a radiation field of energy density $\rho(\omega)$ with an assembly of atoms which induces transitions between energy levels $|1\rangle$ and $|2\rangle$ whose frequency separation is ω_{21}.

The populations in the upper and lower states are N_2 and N_1 respectively, and their degeneracies are g_2 and g_1 respectively. Define the Einstein coefficients A, B_{12} and B_{21} and, by considering the case of thermal equilibrium, find relations between them. [9]

[You may assume that for black body radiation $\rho(\omega)$ is given by the Planck formula:

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1} \ .]$$

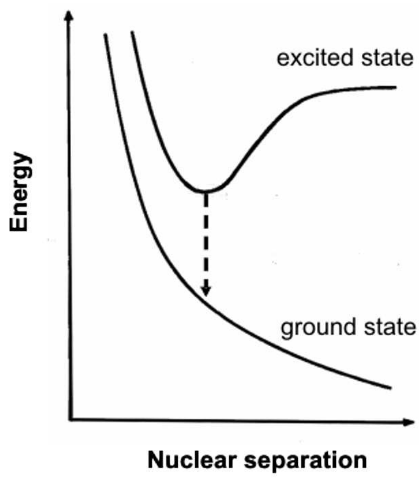
Consider now that a beam of light with intensity $I(\omega, z)$ propagates along the z-axis through a medium containing such atoms. The transition is homogeneously broadened and has a normalised lineshape function $G(\omega - \omega_{21})$. Show that the intensity increases as $\mathrm{e}^{\alpha z}$ where the gain factor α is the product of the inversion density N^* and the optical gain cross-section $\sigma(\omega - \omega_{21})$, where

$$N^* = N_2 - \frac{g_2}{g_1} N_1 \quad \text{and} \quad \sigma(\omega - \omega_{21}) = \frac{\hbar\omega_{21}}{c} B_{21} G(\omega - \omega_{21}) \ .$$ [6]

For the case of lifetime broadening, what is the shape of the gain profile and how does its width depend on the upper and lower state lifetimes? Show that the maximum possible gain cross section is

$$\sigma_{21}(\max) = \frac{2\pi c^2}{\omega_{21}^2} \ .$$ [6]

A schematic diagram showing the energy levels of an excimer laser is given in the figure.



Explain what is unusual about this system and why it makes a good laser. The upper and the ground state lifetimes are approximately 10^{-8} s and 10^{-14} s, respectively, and $\omega_{21}/c = 40,000\,\mathrm{cm}^{-1}$. Calculate the value of the optical gain cross-section. [4]

4. An atom with two levels, whose eigenstates are ψ_1 and ψ_2 with energies E_1 and E_2 respectively, is subjected to an intense polarised light field $\hat{\mathbf{x}}\,\mathcal{E}\cos\omega t$ (where $\hat{\mathbf{x}}$ is a unit vector in the x-direction) at resonance, i.e. $\hbar\omega = E_2 - E_1$. The perturbed Hamiltonian is

$$\hat{H} = \hat{H}_0(\mathbf{r}) + \hat{V}(t)$$

and the general solution to the time-dependent Schrödinger equation can be written

$$\Psi(\mathbf{r},t) = \sum_{i=1}^{2} c_i(t)\psi_i(\mathbf{r})\exp(-\mathrm{i}E_i t/\hbar)\ .$$

(i) Find expressions for V_{ij}, the matrix elements of the perturbation arising from the interaction of the field with the atomic dipole whose matrix elements are $\mathbf{d}_{ij}$. [5]

(ii) Show that

$$\begin{aligned} \dot{c}_1(t) &= \frac{1}{\mathrm{i}\hbar}V_{12}\,c_2(t)\exp\left[-\mathrm{i}(E_2-E_1)t/\hbar\right]\ , \\ \dot{c}_2(t) &= \frac{1}{\mathrm{i}\hbar}V_{21}\,c_1(t)\exp\left[-\mathrm{i}(E_1-E_2)t/\hbar\right]\ . \end{aligned}$$

[6]

(iii) Find expressions for the time-dependent occupation probabilities for both states and show that they display Rabi oscillations with frequency $\Omega_{\mathrm{R}} = \mathcal{E}|\mathbf{d}_{ij}|/\hbar$. [5]

(iv) The light field is a square pulse of intensity I_0 and duration T. What is the condition on I_0 for the pulse to be a π-pulse? Estimate the required intensity for a 2-level atom with $|\mathbf{d}_{ij}| = 10^{-29}\,\mathrm{C\,m}$, and $T = 1\,\mathrm{ns}$. (You may assume the refractive index is 1.) [6]

Illustrate your answers to (iii) and (iv) using the Bloch sphere representation assuming that initially all atoms are in the lower state, i.e. $c_1(0) = 1$, $c_2(0) = 0$. [3]

[Use the convention that the angular co-ordinates (θ,ϕ) on the Bloch sphere are defined by $c_1 = \sin(\theta/2)$ and $c_2 = \mathrm{e}^{\mathrm{i}\phi}\cos\theta/2)$.]

Section IV. (Sub-atomic physics)

5. Define the differential scattering cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ in terms of probabilities and give an expression for it in terms of particle fluxes. [3]

Derive the Lippmann Schwinger equation (LSE) which relates the scattered wave functions $|\psi^{(\pm)}\rangle$ to the un-scattered wavefunctions $|\phi\rangle$, the scattering potential V and the free particle Hamiltonian H_0. [6]

Far from the centre of a finite range potential the LSE, written in position space, simplifies to

$$\langle x|\psi^{(+)}\rangle \approx \frac{1}{(2\pi)^{3/2}}\left[\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} f(\mathbf{k}',\mathbf{k})\right]$$

where the scattering amplitude $f(\mathbf{k}',\mathbf{k})$ is given by

$$f(\mathbf{k}',\mathbf{k}) = -\frac{(2\pi)^4 m}{h^2}\int \frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{x}'}}{(2\pi)^{3/2}} V(\mathbf{x}')\langle \mathbf{x}'|\psi^{(+)}\rangle\, \mathrm{d}^3x'$$

and $\mathbf{k}$ and $\mathbf{k}'$ are the wave vectors of the incoming and scattered particles, $k = |\mathbf{k}|$ and $r = |\mathbf{x}|$. Explain why the solution $\psi^{(-)}$ is not considered any more. [2]

Using the Born approximation and a spherically symmetric potential $V(r)$, show that the scattering amplitude is given by

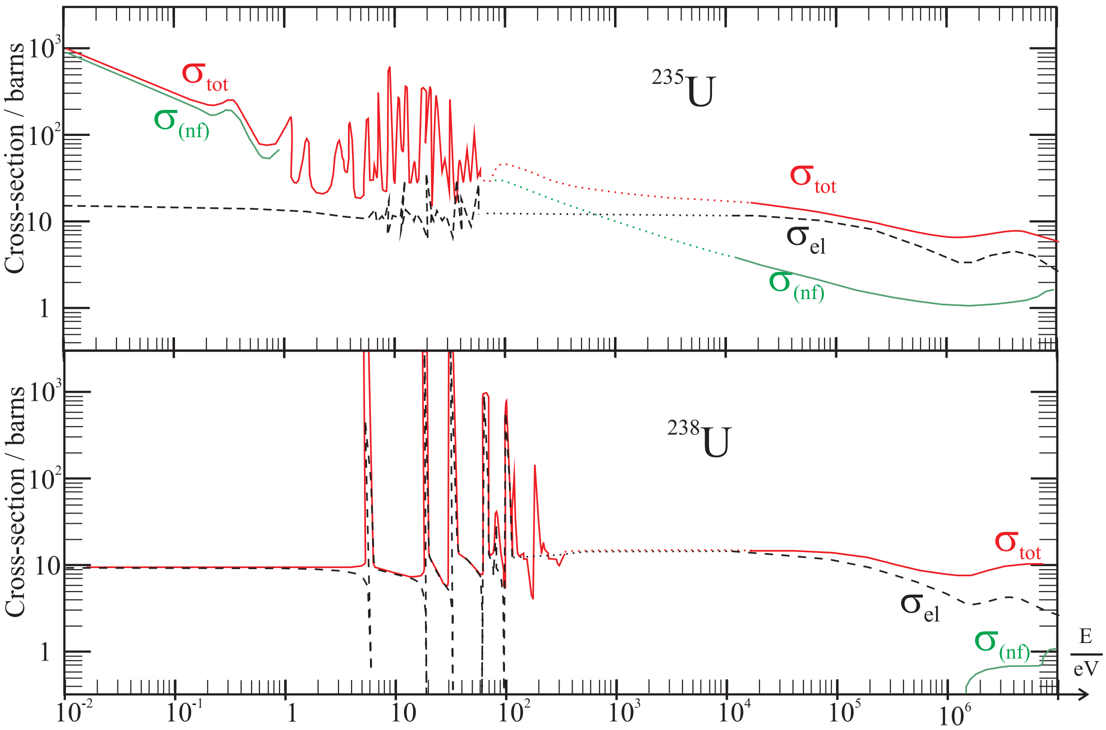
$$f^{(1)} = -\frac{2m}{\hbar^2 q}\int rV(r)\sin qr\, \mathrm{d}r\ ,$$

where $q = |\mathbf{k}-\mathbf{k}'|$. State the relationship between $f^{(1)}$ and the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$. [6]

Use the above result to calculate $\mathrm{d}\sigma/\mathrm{d}\Omega$ as a function of the scattering angle θ, and neutron energy E for the scattering of low energy neutrons by heavy nuclei. The scattering potential can be approximated as an attractive spherical well of depth V_0 and radius R. You may assume that the reduced wavelengths $\lambda\!\!\!^{-} = \lambda/2\pi$ of the neutrons are much larger than R. [8]

6. Explain how energy can be released in both fission and fusion of nuclei. [3]

Sketch the potential energy of the fission fragments of a heavy nucleus versus their separation. Explain qualitatively how the shape arises. Indicate the fission barrier energy E_f and Q-value in your sketch. What does this imply for the stability of heavy nuclei against spontaneous fission? Derive the minimum kinetic energy a neutron must have to induce fission in a heavy nucleus. [6]

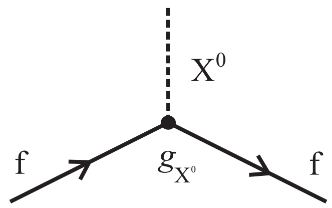


The figure above shows the total, elastic and induced fission cross-sections for ${}^{238}_{92}$U and ${}^{235}_{92}$U versus neutron kinetic energy E. The dotted lines indicate average cross-sections where narrow resonances cannot be resolved. Explain why ${}^{238}_{92}$U has negligible cross-section for neutron-induced fission below approximately 1 MeV whereas $\sigma^{235}_{(n,f)}$ is significant down to 0 MeV. List the typical products of Uranium fission. What enables fission reactors to be controlled on a time-scale of seconds? [6]

Let p be the probability that a neutron with kinetic energy E induces fission in a single collision with a nucleus in natural uranium containing $c = 0.7\%$ of ${}^{235}_{92}$U by number. Let q be the probability that a neutron induces fission in natural uranium *before* being absorbed. You may assume that the only inelastic process for neutrons in the plotted energy range is induced fission. Find expressions for p and q in terms of the cross-sections σ_{nf} (induced fission), σ_{abs} (absorption) and σ_{tot} (total). From the plots obtain a crude estimate for p and q at $E = 2$ MeVand for thermal neutrons at $T = 300\,°$C. [5]

Assuming that fission produces on average ν neutrons find an expression for the neutron multiplication factor η in terms of one of the above probabilities. Compute η for $E = 2$ MeVand for thermal neutrons at $T = 300\,°$C and comment on the implications for a sustained chain reaction. How therefore can a fission reactor use natural uranium as fuel? [5]

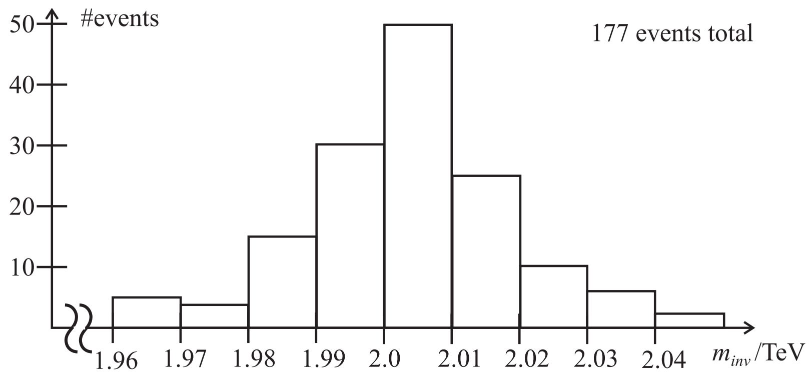
7. Let us assume that scientists at the LHC would discover a new particle X^0 of mass M_{X^0}. The only interactions it has are described by the Feynman diagram below, where g_{X^0} is a universal coupling constant valid for all fermions f.



Draw a parton level Feynman diagram of lowest order in g_{X^0} for the production of the X^0 at the LHC. Make sure to explain how any partons in your diagram connect to the valence quarks of the protons. [3]

If the two constituents of the proton which ultimately produce an X^0 have momentum fraction x_1 and x_2 respectively and if the proton-proton centre of mass energy is $\sqrt{s}$ find an approximate relationship between m_{X^0}, $\sqrt{s}$, x_1 and x_2 that is independent of the proton mass. Clearly state any approximations you make. [5]

The X^0 has been observed in its jet-jet, $\mu^+\mu^-$ and other final states. For an integrated luminosity of 2 fb^{-1} the graph below shows the number of events $n_{jj}(m_{inv})$ per invariant mass interval in which the X^0 decays into a two-jet final state (background was subtracted) as a function of the invariant mass of its decay products m_{inv}.



Derive an expression for the ratio $N_{\mu^+\mu^-}/N_{jj}$ where $N_{\mu^+\mu^-}$ (N_{jj}) are the total number of events you expect to observe in the $\mu^+\mu^-$ (jet-jet) final states. Compare the shape and normalisation of the corresponding graph for $\mu^+\mu^-$ final states to the one shown above. [4]

Write down the functional form of n_{jj} as a function of m_{inv} and the particle spins. Under which assumptions is this form accurate? You may ignore any normalisation constants. Explain the meaning of all terms in your functional form. [5]

Deduce the production cross-section $\sigma(\mathrm{pp} \to X^0)$, the mass and the lifetime of the X^0 from the diagram. You may assume that that the detector which obtained this result is fully efficient and covers the full solid angle around the collision point. [5]

Deduce the Baryon number of the X^0. [3]

8. The oscillations of neutrinos produced in the Earth's atmosphere by cosmic ray interactions can be described approximately as an oscillation between two species. We assume that the weak eigenstates $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ with momentum p, are mixtures of the mass eigenstates $|\nu_2\rangle$ and $|\nu_3\rangle$ with masses m_2 and m_3 and energies E_2 and E_3 respectively. The mixing matrix U is defined as follows:

$$\begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} = U \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} .$$

Let the probability that an originally pure $|\nu_\mu\rangle$ state oscillates into a $|\nu_\tau\rangle$ state after a time t be $P(\nu_\mu \to \nu_\tau, t)$. Starting from the time dependent Schroedinger equation, show that in natural units:

$$P(\nu_\mu \to \nu_\tau, t) = 4\cos^2\theta \sin^2\theta \sin^2\left(\frac{E_3 - E_2}{2}\, t\right) .$$

[6]

Give an expression for the oscillation probability $P(\nu_\mu \to \nu_\tau, L)$ as a function of distance L in km, the squared mass difference $\Delta m^2_{3,2} = m_3^2 - m_2^2$ in $\mathrm{eV}^2/\mathrm{c}^2$ and the average energy $E = (E_3 + E_2)/2$ in GeV. [5]

Draw the lowest order Feynman-diagrams for elastic scattering of $\bar{\nu}_e$ and $\bar{\nu}_\mu$ with electrons. Estimate an approximate range for the ratio of the total elastic scattering cross-sections $R = \frac{\sigma(\bar{\nu}_\mu \mathrm{e} \to \bar{\nu}_\mu \mathrm{e})}{\sigma(\bar{\nu}_e \mathrm{e} \to \bar{\nu}_e \mathrm{e})}$. [5]

Assuming that $\sigma(\bar{\nu}_e \mathrm{e} \to \bar{\nu}_e \mathrm{e})$ is dominated by charged, weak interactions write down the essential factors for the corresponding lowest order amplitude. Find an expression for the cross-section $\sigma(\bar{\nu}_e \mathrm{e} \to \bar{\nu}_e \mathrm{e})$ in terms of your above result for the amplitude. You should use dimensional analysis to find a factor, which is a function of the squared centre of mass energy s only, that gives your result the correct dimension. Assuming that the neutrino energy $E_\nu = 3.7\,\mathrm{MeV}$, find an approximate expression for s in terms of E_ν and hence compute $\sigma(\bar{\nu}_e \mathrm{e} \to \bar{\nu}_e \mathrm{e})$ in barn. [6]

A nuclear reactor produces a rate of $R = 3 \times 10^{20}\ \mathrm{s}^{-1}$ anti-neutrinos per second, with an average energy of $\langle E \rangle = 3.7$ MeV. These anti-neutrinos are detected in a detector containing a liquid scintillator target of mass $m_t = 1000$ tons. It is $r = 10$ km away from the reactor. The scintillator has an electron number density of $\rho_e = 3.4 \times 10^{23}\,\mathrm{g}^{-1}$.

Making the simplifying assumption that anti-neutrinos react only via elastic scattering with electrons in the scintillator and using your result for $\sigma(\bar{\nu}_e \mathrm{e} \to \bar{\nu}_e \mathrm{e})$, calculate the number N of interactions you expect to detect per day, in the absence of neutrino oscillations. Explain how and why N would change if neutrino oscillations were to happen. [3]