2663

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A3: QUANTUM PHYSICS

TRINITY TERM 2011

Friday, 24 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. For a quantum mechanical operator Q, what do the *expectation value* $\langle Q\rangle$ and the *dispersion* $D(Q) = \sqrt{\langle (Q - \langle Q\rangle)^2\rangle}$ represent?

If a system is in a eigenstate of Q with eigenvalue q, evaluate $\langle Q\rangle$ and $D(Q)$ for the system. [4]

2. What is the *parity* operator in quantum mechanics and what are the parity eigenvalues of the eigenstates under its action?

If ψ_1 and ψ_2 are *degenerate* eigenstates of a quantum-mechanical Hamiltonian, what does this imply about them?

Show that the non-degenerate eigenstates of a Hamiltonian which is symmetric under space inversion of the coordinates have definite parity. Comment on the degenerate case. [5]

3. How is the *probability current density* related to the static probability density?

Considering a particle moving in one dimension in a potential $V(x)$, derive an expression for the probability current density j, starting from the time-dependent Schrödinger equation. [6]

4. By considering the time-independent Schrödinger equation, comment on the general form of the wavefunction ψ in regions where $E < V$ and in regions where $E > V$, taking V to be a constant. If a particle of energy E travels from the region $x < 0$, for which $V = 0$, into the region $x > 0$, for which $V = V_0 > E$, what is the probability that the particle will be reflected? What is the probability to observe the particle at distance $x = d$ ($d > 0$), relative to the probability to observe it at $x = 0$? [7]

5. At time $t = 0$, a free particle of mass m moving in one dimension is described by the normalised wavefunction

$$\psi(x) = \left(\frac{2}{\pi a^2}\right)^{1/4} \exp\left(-\frac{x^2}{a^2} + ikx\right).$$

Obtain the expectation values of the position and momentum of the particle and estimate the size of the region in which it is located. Show that the probability current density can be written as $(\hbar k/m)|\psi(x)|^2$. [9]

6. For a two-state system, the time-independent Schrödinger equation, $H|\psi\rangle = E|\psi\rangle$, can be written as an eigenvalue equation for the energy E:

$$\begin{pmatrix} E_1 + \lambda & \lambda \\ \lambda & E_2 + \lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix}.$$

What are the eigenvalues of H when $E_1 = E_2$ and what are the associated eigenstates of the system?

Comment on how your answer relates to first-order perturbation theory. [9]

Section B

7. Hydrogen-like wavefunctions have the form

$$\psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta, \phi).$$

Which property of the Coulomb interaction allows for the above separation of variables? What do the symbols n, ℓ and m represent? For a given n, what values can ℓ and m take? [5]

The ground-state wavefunction is given by

$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2\,\mathrm{e}^{-Zr/a_0} \qquad \text{and} \qquad Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}\ ,$$

where $a_0 = 4\pi\epsilon_0\hbar^2/\mu e^2$. What do μ and Z represent? What is the significance of a_0? Sketch the probability density for the electron as a function of r. [5]

A tritium atom, ^{3}H is in its ground state when the nucleus undergoes a beta decay and becomes ^{3}He. Using the 'sudden approximation' calculate the probability that this helium ion is in the $1s$ state. [10]

8. J_x, J_y and J_z, are the angular momentum operators. Evaluate the commutators $[J_z, J_+]$, $[J_z, J_-]$, $[J^2, J_+]$, and $[J^2, J_-]$, where $J^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x \pm iJ_y$. [3]

If $|j, m\rangle$ is an eigenket of J^2, J_z with eigenvalues $j(j+1), m$ show that $J_+|j, m\rangle$ and $J_-|j, m\rangle$ are also eigenkets and give the corresponding eigenvalues. [3]

Derive matrices representing J_x, J_y, J_z for $j = 1$. [8]

The matrix for the angular momentum in a direction at 45° to the z axis in the xz plane is:

$$J_{45^\circ} = \frac{1}{\sqrt{2}}(J_x + J_z).$$

Find the eigenvector of this operator corresponding to the eigenvalue zero. [3]

A spin-1 particle is prepared in a state with magnetic quantum number $m = 0$ with respect to a reference axis at 45° to the z axis in the xz plane. It then passes through a filter which only transmits particles in the $m = 1$ state with respect to the z direction. What is the probability that the particle is still in the $m = 0$ state with respect to the original 45° axis? [3]

9. A particle is confined in an infinitely deep square well described by a potential $V(x)$ which is zero for $-\frac{1}{2}a \le x \le \frac{1}{2}a$. Use the Schrödinger equation to find the energies available to the particle and sketch the wavefunctions of the three states of lowest energy. [8]

Explain how the wavefunction of the ground state is changed if the well is perturbed by a small added potential V':

(a) $V' = V_1 x$ $\quad$ ($V_1 > 0$, $V_1 a \ll E_0$, the ground-state energy),

(b) $V' = V_2 x^2$ $\quad$ ($V_2 > 0$, $V_2 a^2 \ll E_0$). [5]

The well is perturbed by the addition to $V(x)$ of a potential $V' = V_3 \sin(\pi x/a)$. Show that in perturbation theory to first order, only the first excited-state wavefunction then mixes with the ground-state wavefunction and evaluate its relative contribution. [7]

10. The Hamiltonian for the interaction between an electron and an uniform magnetic field $\vec{B}$ is

$$H = \frac{e\hbar}{m_{\mathrm{e}}}\vec{B}\cdot\vec{S},$$

where $\vec{S}$ is the electron spin with components:

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The magnetic field (of magnitude B) is along the z-axis. Show that the wavefunction for the electron takes the form

$$\psi(t) = \begin{pmatrix} a(0)\exp\left(-\frac{1}{2}\,\mathrm{i}\,\omega_L t\right) \\ b(0)\exp\left(\frac{1}{2}\,\mathrm{i}\,\omega_L t\right) \end{pmatrix}, \text{ where } \omega_L = eB/m_{\mathrm{e}}$$

and specify the energy eigenvalues. [6]

A magnetic field of magnitude $B' \ll B$, which rotates in the xy plane at angular velocity ω, is applied to the system in the lower eigenstate. What is the probability that it will be found in the higher energy eigenstate at time t? [14]