2661

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A1: THERMAL PHYSICS

TRINITY TERM 2011

Wednesday, 22 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. An engine is operated between a body of heat capacity C_1 and temperature T_1 and a second body of heat capacity C_2 and temperature T_2, in order to extract work. You may assume that C_1 and C_2 are independent of temperature and that $T_1 > T_2$. Assume also that the engine produces the maximum possible amount of work.

Derive expressions for the final temperature T_f of the bodies and for the work performed by the engine. [7]

2. A thin-walled container of volume $10^{-3}\,\mathrm{m}^3$ is evacuated to a pressure of 10^{-7} mbar. The vessel is surrounded by air at 1 bar and 290 K. If a small hole of area $10^{-17}\,\mathrm{m}^2$ is made in the wall of the container, how long does it take the pressure inside to rise to 10^{-6} mbar?
[1 bar = 10^5 Pa; 1 mbar = 10^{-3} bar. You may assume that the mean speed of molecules in a gas at temperature T is $(8k_\mathrm{B}T/\pi m)^{1/2}$, where m is the molecular mass.] [8]

3. The temperature inside a house is 290 K. The owner turns the central heating up so that the temperature becomes 291 K. What is the increase in the total energy of the air inside the house following this change of temperature? [4]

4. Estimate the energy density of thermal radiation in equilibrium inside a container whose walls are held at a temperature of 300 K. [4]

5. The energy levels of a system consist of a ground state level (energy $E = 0$) and a triply–degenerate excited state (energy $E = \Delta$). Derive expressions for $\langle E\rangle$ and $\mathrm{Var}E = \langle E^2\rangle - \langle E\rangle^2$ when in equilibrium with a reservoir at temperature T and show how your expressions behave in the limits $T \to 0$ and $T \to \infty$. [9]

6. The gradient of the melting line of water on a p–T diagram close to $0\,^\circ\mathrm{C}$ is $-1.4 \times 10^7\,\mathrm{Pa\,K^{-1}}$. At $0\,^\circ\mathrm{C}$, the specific volume of water is $1.00 \times 10^{-3}\,\mathrm{m^3\,kg^{-1}}$ and of ice is $1.09 \times 10^{-3}\,\mathrm{m^3\,kg^{-1}}$. Using this information, deduce the latent heat of fusion of ice, expressing your result in $\mathrm{MJ\,kg^{-1}}$. [4]

7. The functions $f_n(x)$, where n is any non-negative integer, satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathrm{e}^{-x}\,\frac{\mathrm{d}f_n}{\mathrm{d}x}\right) + n\mathrm{e}^{-x}f_n = 0 \quad \text{for} \quad 0 \le x < \infty\,.$$

Each f_n and its derivative is bounded at $x = 0$ and as $x \to \infty$. Show that, if m is a non-negative integer not equal to n, then

$$\int_0^\infty \mathrm{e}^{-x} f_n(x) f_m(x)\,\mathrm{d}x = 0\,.$$

[4]

Section B

8. In a Joule expansion, a gas in volume V_1 at temperature T_1 undergoes free expansion to a larger volume V_2 and final temperature T_2. The whole process takes place under thermally isolated conditions. Show that in such a process the internal energy U of the gas is conserved. [3]

Show that the Joule coefficient is

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V}\left[T\left(\frac{\partial p}{\partial T}\right)_V - p\right].$$

[5]

The equation of state for one mole of a van der Waals gas is

$$p + \frac{a}{V^2} = \frac{RT}{V-b}\ .$$

Derive an expression for the cooling $\Delta T = T_1 - T_2$ for a van der Waals gas undergoing a Joule expansion from volume V_1 to V_2. [You may assume that C_V is independent of temperature.] Consider the limits (i) $a = 0$, $b \neq 0$ and (ii) $b = 0$, $a \neq 0$, and discuss the physical reasons for the value of ΔT in each case. [7]

Explain why the Joule expansion is not a practical cooling mechanism for liquefying gases, and outline the principles of a gas liquefier based upon an alternative mechanism. [5]

9. The energy E of a three-dimensional harmonic oscillator is given by

$$E = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega + \left(n_z + \frac{1}{2}\right)\hbar\omega.$$

Show that the partition function Z of this system is given by $Z = Z_{\mathrm{SHO}}^3$, where

$$Z_{\mathrm{SHO}} = \frac{\mathrm{e}^{-\frac{1}{2}\beta\hbar\omega}}{1 - \mathrm{e}^{-\beta\hbar\omega}},$$

[5]

and $\beta = 1/(k_{\mathrm{B}}T)$. Show that the Helmholtz function of this oscillator is given by

$$F = \frac{3}{2}\hbar\omega + 3k_{\mathrm{B}}T\ln(1 - \mathrm{e}^{-\beta\hbar\omega}),$$

and find expressions for the entropy, the internal energy and the heat capacity. [11]

Show that the heat capacity tends to a constant value at high temperature and show that this is in agreement with the equipartition theorem. [4]

10. Derive an expression for the partition function of a classical gas of N spinless indistinguishable particles of mass m in a volume V at temperature T. [5]

Show that for such a gas the entropy S is given by

$$S = Nk_\mathrm{B}\left[\alpha - \ln\left(\frac{N}{V}\lambda(T)^3\right)\right],$$

where α is a numerical constant and $\lambda(T)$ is a function of temperature, both of which you should find. Use the expression for S to show that for an ideal monatomic gas under adiabatic conditions the pressure p obeys the law

$$pV^{5/3} = \text{constant}.$$ [8]

Two equal volumes of ideal gases at the same temperature and pressure are mixed. Find the entropy change (a) when the gases are identical and (b) when they are different. Comment on your answer. [7]

11. The Fourier transform $\tilde{f}(k)$ of the function $f(x)$ is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} f(x).$$

Write down the inverse Fourier transform. Show that

(a) the Fourier transform of $\mathrm{d}f/\mathrm{d}x$ is $\mathrm{i}k\tilde{f}$

(b) the Fourier transform of $\mathrm{e}^{\mathrm{i}qx}$ is $2\pi\delta(k-q)$

Find an expression for the Fourier transform $\tilde{y}(k)$ of the function $y(x) = \cos(qx)$. [10]

The thermal diffusion equation is given by

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}.$$

By Fourier transforming this equation, show that the solution is given by

$$\tilde{T}(k,t) = \tilde{T}(k,0)\mathrm{e}^{-Dk^2 t},$$

and hence find $T(x,t)$ for the initial condition

$$T(x,0) = T_0 + \sum_{m=1}^{\infty} T_m \cos\left(\frac{m\pi x}{L}\right),$$

where T_m $(m = 0, 1, 2\cdots)$ are constants. Show that for $t \gg L^2/D$ the solution is approximately given by

$$T(x,t) = T_0 + T_1\mathrm{e}^{-D\pi^2 t/L^2}\cos\left(\frac{\pi x}{L}\right).$$

[10]