FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SECOND PUBLIC EXAMINATION

Honour School of Physics,
Parts A and B: 3 and 4 year Courses

SHORT OPTIONS

TRINITY TERM 2011

Tuesday, 21 June
9.30 am to 11.00 am for candidates offering ONE Short Option
9.30 am to 12.30 pm for candidates offering TWO Short Options

Answer **two** *questions from each option for which you have entered.*

Start the answer to each question in a fresh book.

If you have entered for two Short Options,
keep your answers to the two options in different books
and at the end hand in two bundles, one for each option.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate
assigning to each part of the question.

Do NOT turn over until told that you may do so.

INDEX

Some sections start with a relevant rubric.

Section S1   FUNCTIONS OF A COMPLEX VARIABLE

1. (a) Solve the following equation in the complex z plane:

$$\sin z = 3 .$$

[8]

(b) Consider the Laurent series expansion about $z = 0$ of the function

$$f(z) = \frac{\sin z}{z^4} .$$

Determine the principal part of the series and the first term of the analytic part of the series.

[8]

(c) Evaluate the following real integral by complex contour integration methods:

$$\int_0^{2\pi} \frac{1}{2 + \cos t} \, \mathrm{d}t .$$

[9]

2. Consider the function

$$u(x, y) = e^{-x} \cos y + xy .$$

(a) Show that $u(x, y)$ is harmonic over the whole xy plane.

[4]

(b) Find a holomorphic function $f(z)$ whose real part is given by $u(x, y)$ and which is real-valued at $z = 0$ ($z = x + \mathrm{i}y$).

[7]

(c) Write down the family of curves in the xy plane which are orthogonal to the curves

$$e^{-x} \cos y + xy = \text{constant} .$$

[7]

(d) Evaluate the integral of $u(x, y)$ round the circle in the xy plane with centre at the origin and radius 1.

[7]

3. (a) Give the location and order of the branch points of the function

$$f(z) = \frac{\ln(z + \mathrm{i})}{1 + z^2} .$$

[4]

(b) Take the principal branch of the logarithm and evaluate the integral of $f(z)$ round the closed contour Γ_R

$$\oint_{\Gamma_R} \mathrm{d}z \; f(z) ,$$

where Γ_R consists of the semicircle in the upper half plane centred at $z = 0$ with radius R, $R > 1$, and of the real-axis interval $(-R, R)$.

[9]

(c) Use the result in (b) to calculate the real integral

$$\int_{-\infty}^{\infty} \frac{\ln(1 + x^2)}{1 + x^2} \, \mathrm{d}x .$$

[12]

Section S2   ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. State Kepler's third law of planetary motion and write it in equation form in either S.I. units or other convenient units which you should define.   [5]

A star of mass M_S exhibits a sinusoidal variation in its radial velocity of amplitude V_S and period P arising from the motion induced by a planetary companion. By assuming that the orbit of the planet is circular, edge-on to the observer and that the mass of the planet, M_P, is small compared to the star, show that

$$M_P^3 \simeq V_S^3 \frac{PM_S^2}{2\pi G} \ .$$

    [12]

What velocity does the Earth induce in the Sun? Would such a motion be detectable in a star other than the Sun using current technology? What other method do we use to detect Earth-mass planets around solar type stars?   [8]

[The mass of Earth is $6 \times 10^{24}\,\mathrm{kg}$.]

2. Describe with the aid of diagrams and/or equations **three** ways of estimating the surface temperature of a star which do not require the distance to the star to be known.   [10]

A star of apparent bolometric magnitude m_{bol}=9.3 has a measured parallax of 0.033 arcseconds and its temperature is estimated to be 3500 K. By assuming the star radiates like a blackbody, estimate its radius in units of solar radii.   [10]

The star is primarily composed of hydrogen but the spectrum of this star exhibits strong absorption lines arising from atomic manganese, iron and silicon and molecular titanium oxide. Why are the Balmer lines of hydrogen absent from the spectrum?   [5]

[Absolute bolometric magnitude of the sun: $M_{\odot bol} = 4.72$.]

3. Write down the equation of the Virial Theorem for a dynamical system in equilibrium, defining the terms used.   [5]

A cluster of galaxies contains 2000 galaxies within a sphere of 1 Mpc diameter. The root-mean-square velocity of the galaxies is measured to be 1100 $\mathrm{km\,s^{-1}}$ and the rotation of the cluster is negligible. Make an estimate of the total mass of the cluster in $M_\odot$. By assuming each galaxy has a luminosity of $5 \times 10^9 L_\odot$, and that on average $2\,M_\odot$ generates $1\,L_\odot$ make an estimate of the mass of the galaxies in the cluster. Give two possible explanations for these results; which is preferred and why?   [13]

Describe qualitatively two other methods for determining the total mass of a cluster of galaxies.   [7]

Section S3   QUANTUM IDEAS

1. What is so puzzling about the photoelectric effect if one tries explaining it with a continuous theory of electromagnetic radiation, and how did Einstein resolve these issues using the concept of photons?   [4]

What is the momentum and wavelength of a photon with an energy of $0.1\,\text{eV}$ (infrared) and $2\,\text{eV}$ (visible)? Compare the photon wavelength to the de Broglie wavelength of neutrons having kinetic energies identical to these photon energies. Why are these wavelengths so different?   [5]

A slit of width b is illuminated by a parallel beam of electrons with kinetic energy $E_k = 2\,\text{eV}$. Given an observation screen placed at a distance D from the slit, for what value of b does one observe a central maximum of width B (measured from minimum to minimum)? How does the width of the central maximum change if one illuminates this slit with photons of (a) same wavelength, or (b) same energy.   [8]

Show that, by considering conservation of energy and momentum, a free electron cannot absorb a photon. Why is it possible to conserve both energy and momentum in a resonant atomic absorption, and approximately how many photon absorptions at $\lambda = 780\,\text{nm}$ would one need to bring a rubidium atom ($m = 1.4 \cdot 10^{-25}\,\text{kg}$) travelling at an initial velocity of $v_0 = 100\,\text{m/s}$ to a complete standstill?   [8]

2. Discuss what happens to an electron orbiting a nucleus in an all-classical world, and state the assumptions made by Bohr to resolve this problem. In particular, what is the constraint these assumptions impose on the de Broglie wavelength of the electron?   [4]

Starting from Bohr's assumptions, derive expressions for the first Bohr radius of the electron and its associated kinetic energy. Find their numerical values for (a) the He$^+$ ion ($Z = 2$) and (b) a single electron bound to a gold nucleus with $Z = 79$. Compare your kinetic-energy values to the respective minimum energy that results from applying the uncertainty relation to electron orbits of finite size.   [6]

Calculate the wavelength of the Lyman-α line (the transition from the first excited state to the ground state) for (a) hydrogen atoms and (b) positronium e^+e^-, and explain the origin of their difference.   [7]

In the light emitted from a star, a pattern of emission lines is found with every second line coinciding with a line of the Lyman series of hydrogen. Identify which ion causes this pattern and identify the observed transitions [assume the star is stationary with respect to the Earth].   [8]

3. Write down the time-dependent Schrödinger equation (TDSE) for a particle of mass m free to move only in the x direction in a region where the potential is $V(x)$. Explain briefly the meaning of each term. [3]

Assume the potential is constant, $V(x) = V_0$, and make the plane-wave *Ansatz* $\Psi(x, t) \propto \exp(ikx - i\omega t)$ for the wavefunction of the particle. Use the Schrödinger equation to determine whether the angular frequency ω is proportional to the kinetic, total or potential energy of the particle. [6]

How many energy levels of a particle with mass m fit into a rectangular potential well with width a up to a given maximum energy E_{max}, if the bottom of the well is at $V = 0$ and the walls are infinitely high? For $E_{max} = 10\,\text{eV}$ and $a = 0.7\,\text{nm}$, calculate values for (a) an electron and (b) a proton. [6]

Given that the wavevector k of a photon in a cavity of length a is subject to the same conditions as the wavevector of a massive particle in the above potential, what is the lowest possible photon energy? [3]

For an electron prepared in an equal-amplitude superposition of the ground state and the first excited state of the above potential well, calculate the probability of detecting the electron in the left half of the well. Determine the frequency at which this probability oscillates in time. [7]

[hint: $\int_0^{\pi/2} \sin(x) \sin(2x) dx = 2/3$]

Section S4   ENERGY STUDIES

1. Draw a diagram of a silicon photocell connected to a resistor R, showing the cell's construction. Explain the physical principles of its operation and why the photocell current I_C has the form:

$$I_C = I_L - I_S[\exp(eV/k_B T) - 1] \,,$$

where V is the voltage across the cell caused by the photocurrent I_C through R. Define I_L and I_S.   [9]

What factors limit its efficiency to $\sim 20\%$?   [5]

Give **two** advantages and **two** disadvantages of solar power.   [4]

A solar farm consisting of fixed silicon photovoltaic panels is located at a latitude of 30°, where the average number of hours of direct sunlight is 8 per day. The panels have an efficiency of 20% and a total active area of $2 \times 10^4 \, \text{m}^2$. Estimate the maximum average power output of the solar farm.   [7]

[The intensity of direct sunlight is $1 \, \text{kW} \, \text{m}^{-2}$.]

2. Describe the physical principles of a pressurised water reactor (PWR), explaining why enriched fuel and a moderator are required, and how the reactor is controlled.   [10]

Explain how a *breeder* reactor works.   [4]

A $1.5 \, \text{GW}$ breeder reactor contains enriched uranium fuel rods surrounded by ^{232}Th. For an initial amount of ^{235}U of $1200 \, \text{kg}$ and a breeding ratio of 1.1 to breed ^{233}U, estimate the time to double the amount of fissile material.   [5]

Compare and contrast the advantages and disadvantages of nuclear and wind power.   [6]

[The energy release per neutron-induced fission of a fissile nucleus is $200 \, \text{MeV}$.]

3. Comment *critically* with numerical estimates, where appropriate, on **three** of the following statements:

(a) Reducing the demand for energy is essential if we are are to tackle climate change.

(b) The development of better energy storage devices is required if we are to realise the potential of renewable energy resources.

(c) It is important that research into fusion power is continued to provide a long term solution to the global energy demand.

(d) Biofuels should not be developed as they require too much land area.

(e) The finite size of the global fossil fuel reserves will limit global warming and reduces the need for urgent action.

(f) The installation of heat pumps into all buildings in the UK would alone make a substantial reduction to UK carbon emissions.   [25]

Section S7   CLASSICAL MECHANICS

1. N particles, having masses m_i and positions $\mathbf{x}_i$, move in an externally applied force field that imparts a constant, uniform acceleration $(0,0,g)$ to each particle. In addition, each pair (i,j) of particles has an interaction potential energy $V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$ that depends only on the particles' separation, with $V_{ij} = V_{ji}$.

Using Newtonian mechanics, write down the equations of motion for this system. Explain why solutions to these equations are extremals of a particular action integral, which you should give. (You should state clearly any results you use from the calculus of variations, but do not need to provide proofs.)   [12]

Prove Noether's theorem: if a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is invariant under a transformation $\mathbf{q} \to \mathbf{q} + \epsilon \mathbf{K}(\mathbf{q})$, then the quantity $C = \mathbf{K} \cdot \partial L / \partial \dot{\mathbf{q}}$ is a constant of motion.   [5]

Using Noether's theorem, or otherwise, find four constants of motion for the system above.   [8]

2. A particle is confined to move on the surface of a sphere whose radius $a(t)$ varies with time. Show that a Hamiltonian for this system is

$$H = \frac{p_\theta^2}{2a^2} + \frac{p_\phi^2}{2a^2 \sin^2 \theta} \, ,$$

where (p_θ, p_ϕ) are the momenta conjugate to the polar co-ordinates (θ, ϕ).   [8]

Find two constants of motion, one of which is a multiple of H.   [10]

Hence, or otherwise, show that a particle launched from the equator $\theta = \pi/2$ with velocity $(\dot{\theta}, \dot{\phi}) = (\dot{\theta}_0, \dot{\phi}_0)$ is confined to the region

$$\sin^2 \theta \geq \frac{\dot{\phi}_0^2}{\dot{\theta}_0^2 + \dot{\phi}_0^2} \, .$$

Give a physical reason why this is independent of the form of $a(t)$.   [7]

3. Explain what is meant by the terms *extended phase space*, *Poincaré integral invariant* and *canonical map* in the context of Hamiltonian mechanics. [4]

Starting from conservation of the Poincaré integral invariant, explain how one can construct canonical maps $(\mathbf{q}, \mathbf{p}) \rightarrow (\mathbf{Q}, \mathbf{P})$ by means of a generating function of the form $F_2(\mathbf{q}, \mathbf{P}, t)$, in terms of which the transformation is given implicitly by

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}, \quad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}} \ .$$
[8]

Find the mapping produced by the generating function

$$F_2(q, P, t) = qP \sec \alpha - \frac{1}{2}(q^2 + P^2) \tan \alpha \ ,$$

where $\alpha = \alpha(t)$. [5]

A system with Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ is transformed under this mapping with $\alpha(t) = t$. Find the Hamiltonian $K(Q, P, t)$ for the system in the new variables. Give a physical interpretation of your result. [8]

Section S9   FINANCIAL PHYSICS

1. Describe and compare forward contracts, European options and American options.   [8]

Derive the Black-Scholes equation, explaining carefully the assumptions and approximations that you make. Describe the resulting hedging strategy for the price of a European call option. Discuss how the hedging strategy and the value of the call option vary as the option approaches expiry.   [17]

2. Discuss the mechanics of financial markets including the order of events that lead to the determination of a price.   [8]

In game A the probability of winning at time t is determined by success (in any game) at the previous two timesteps $t-2$ and $t-1$. A win (W) earns one unit of cash, and a loss (L) results in paying one unit of cash. Following a sequence of outcomes (L,L) at time steps $(t-2,\ t-1)$, the probability of winning at timestep t is p_1. Following (L,W) it is p_2, following (W,L) it is p_3 and following (W,W) it is p_4. Let $\pi_1(t)$ be the probability of the sequence (L,L) at timesteps $(t-1,\ t)$, $\pi_2(t)$ be the probability of (L,W), $\pi_3(t)$ be the probability of (W,L), and $\pi_4(t)$ be the probability of (W,W). Find expressions for the π_i in the steady state, for $i = 1$ to 4. Show that a player loses on average when

$$p_1 p_2 < (1-p_3)(1-p_4)\,.$$

[9]

For game B the probability of winning (one unit of cash) is $p_{\mathrm{B}} = 1/2$, in any timestep. This is equal to the probability of losing (one unit of cash). A player switches randomly between the two games A and B. What is the condition that determines whether the player loses on average in this situation? For the case in which $p_2 = p_3 = \frac{1}{4}$ and $p_4 = \frac{7}{10}$ find the range of values of p_1 for which the player wins on average but would lose, on average, if playing only game A. Explain this phenomenon.   [8]

3. What is meant by the financial term *volatility*? Under certain conditions, the volatility of price increments over n timesteps, σ_n, can be related to the volatility over a single timestep σ, by the expression $\sigma_n = n^{1/2}\sigma$. Give the derivation of this expression, stating clearly the conditions under which it will hold true. In a particular market, the price data are found to satisfy the generalised relationship $\sigma_n = n^\alpha \sigma$ with $\alpha > 1/2$. Comment briefly on what this indicates. [11]

In a simple 'coin-toss' market model the probability of a gain (upward price increment) is $p = 1/2$ at each timestep and is independent of previous outcomes. The probability of a downward change is the same: $1 - p = 1/2$. Consider shares with a starting price of $x_0 = 100$ currency units, which may rise or fall by 10% at each timestep. What are the possible share prices after three time steps? What are the probabilities of these outcomes? What is the expectation value of the price? [5]

We purchase a European put option with strike price 110 currency units at timestep 3. What is the payoff of the option for each of the possible final share prices? What is the expectation value of the payoff? [5]

In a market the expectation of the risky return rate and the risk-free return rate are equal. What interest rate is consistent with $p = 1/2$? Suppose instead there is a risk-free investment available that pays 5% interest at each timestep. What value of p does this imply in the market? [4]

Section S10   MEDICAL AND HEALTH PHYSICS

1. Derive the wave equation for a plane sound wave of longitudinal polarisation propagating in a uniform fluid of density ρ and bulk modulus K_B. In the ultrasound imaging of water-like tissue it is required to resolve variations in density or modulus on a mm-scale. Estimate the typical frequency required. [For water, $K_B = 2.2 \times 10^9$ Pa.]   [10]

Describe how waves of such frequency may be generated and how they may be detected to form images.   [7]

Explain why such images suffer from imperfections and what, if anything, can be done to improve them.   [8]

2. Explain the importance of functional imaging with ionising radiation in clinical medicine as an extension of anatomical imaging. Describe how suitable sources of radiation are chosen.   [10]

Draw carefully labelled diagrams showing how, (a) a SPECT scan and (b) a PET scan, may be carried out. Give the particular advantages of each.   [9]

A member of the general public questions the safety of these procedures. Indicate briefly three simple points that you would make to explain the scale of the associated health risks.   [6]

3. A parallel beam of ionising radiation enters the tissue of a patient. Describe briefly the principal mechanisms by which the energy of the beam is lost to the tissue

(a) where the radiation is composed of gamma rays;

(b) where it is composed of charged particles.   [14]

Describe the significance of the following in determining what happens to the biological damage caused by the deposited energy: *apoptosis, single strand breaks, antioxidants.*   [6]

Explain why electron beams are never used for medical imaging and only under particular circumstances for radiotherapy.   [5]

Section S12   INTRODUCTION TO BIOPHYSICS

1. Describe and discuss the structure of the nucleic acids DNA and RNA, their functions in living cells, and ways in which their structure relates to their function.   [25]

2. A synthetic biologist plans to use a single Na^+-coupled F_1F_O-ATP-synthase as a rotary motor to propel a swimming phospholipid vesicle. Rotation is to be driven by hydrolysis of ATP in the medium outside the vesicle. In each rotation, the ATP-synthase tightly couples hydrolysis of 3 molecules of ATP to pumping of 11 Na^+ ions out of the vesicle. The vesicle is 100 nm in diameter and has in its membrane one molecule of ATP-synthase and no other proteins or ion channels.

The ATP-synthase molecule is switched on at time $t = 0$, when the membrane voltage is zero and Na^+ concentrations are 50 mM both inside and outside. Write an expression for the sodium-motive force in the above vesicle in terms of the internal and external Na^+ concentrations and voltages, and the charge q of a Na^+ ion. Hence show that the change in free energy of Na^+ ions when one ion is pumped out of the vesicle is given by

$$\Delta G_{Na^+} = \frac{nq^2}{C} - k_{\mathrm{B}}T \ln\left(1 - \frac{n}{n_0}\right) \ ,$$

where n is the number of ions pumped since $t = 0$, C is the membrane capacitance, n_0 is the total number of ions initially in the vesicle, k_{B} is Boltzmann's constant and T is absolute temperature.   [6]

After N revolutions the ATP-synthase molecule stops because an equilibrium is reached between the sodium-motive force and ATP hydrolysis. Calculate C, and hence write down the number of Na^+ ions that would need to be pumped to establish a membrane voltage equal to the sodium-motive force at the equilibrium above. Compare this to the total number of ions initially in the vesicle, and hence estimate N. You may assume that the vesicle membrane has a thickness of 5 nm and a relative dielectric constant of 2.5, that the free energy of hydrolysis of ATP is 22 $k_{\mathrm{B}}T$, and that the experiment takes place at room temperature.   [12]

Make separate sketches of the electrical and chemical potential components of ΔG_{Na^+} and of the combined ΔG_{Na^+}, all versus n in the range $0 < n < n_0$. Indicate approximately on your sketch of the combined ΔG_{Na^+} the point corresponding to the equilibrium described above.   [7]

3. Fluorescence excitation in Total Internal Reflection Fluorescence (TIRF) microscopy is achieved by illuminating an aqueous sample on the surface of a cover-glass with light travelling through the glass at an angle $\theta > \theta_c$ to the normal, where θ_c is the critical angle. By writing the electric field in the aqueous medium as

$$E = E_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \; ,$$

where E_0 is an electric field amplitude, ω and t are angular frequency and time, $\mathbf{r} = (x, y)$ is a position vector and $\mathbf{k} = (k\sin\theta_2, k\cos\theta_2)$ is a complex wave-vector, derive an expression for $E(x, y, t)$ in terms of θ, ω, k, and the refractive indices of the medium (n_m) and the glass (n_g). [8]

Explain the advantages of this method for observing single fluorescent molecules, and give an expression for a characteristic depth of penetration of the illumination into the medium. [6]

An enzyme is immobilized on the coverglass in a TIRF microscope and a low concentration of fluorescently labelled substrate molecules are added to the medium. It is known that the fluorescent labels are unaffected by enzyme binding and subsequent modification of the substrate, that all binding events lead to irreversible catalysis, and that photobleaching is negligible. Transient bright spots corresponding to single substrate molecules bound to enzymes at the surface are observed. The typical fluorescence intensity of a spot vs time is shown below.
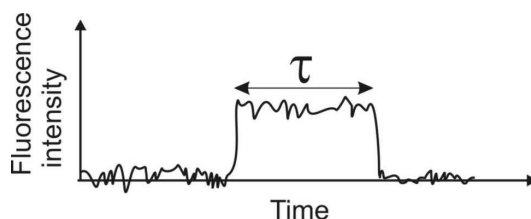
The distribution of time intervals is

$$P(\tau) = \frac{k_a k_b}{k_a - k_b}\left(e^{-k_b\tau} - e^{-k_a\tau}\right).$$

What can be inferred from this observation about the catalytic mechanism of the enzyme? Your answer should include a derivation and a sketch of the distribution $P(\tau)$. [11]

Section S16   PLASMA PHYSICS

1. A hydrogen plasma contains n_e electrons per unit volume at a temperature T_e. Derive expressions for the plasma frequency, ω_p, the Debye length, λ_D, and the plasma parameter, N_D. Demonstrate that for N_D large compared with unity, the average kinetic energy of the electrons is large compared with their average Coulomb energy.   [8]

Show that for light of angular frequency ω the refractive index of the plasma, μ, is given by

$$\mu = \sqrt{1 - \left(\frac{n_e e^2}{\epsilon_0 m_e \omega^2} \right)} \ .$$

[4]

A ray of light with angular frequency ω travelling in vacuo in the xy plane is incident upon a plasma at an angle of incidence θ as shown in the diagram.
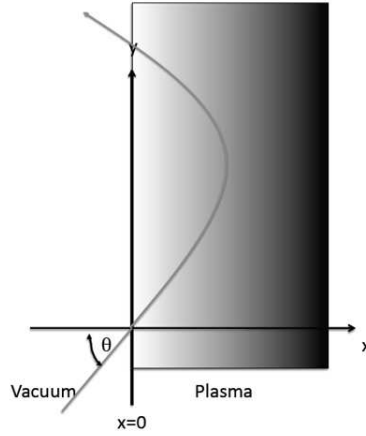
The plasma is confined to the region $x \geq 0$. The number density of the electrons, $n_e(x)$ is uniform in the y direction, but increases monotonically from zero with increasing x. Show that the light is reflected from a point within the plasma, x_R, where

$$n_e(x_R) = \frac{\epsilon_0 m_e \omega^2 \cos^2 \theta}{e^2} \ .$$

[5]

A parallel beam of light of diameter 1 m from a high power optical laser is focussed by a lens onto a cryogenic solid hydrogen target. During the pulse the absorption of the laser light heats the surface of the target, creating a fully ionized plasma that streams away from the surface of the solid, with the plasma density falling off monotonically as a function of distance. The focussed light is reflected at points within the plasma ranging in electron number density from 3.6×10^{21} to 4.0×10^{21} cm^{-3}. What is the wavelength of the laser, and the focal length of the lens?   [4]

Assuming that the electron temperature of the plasma is 300 eV, without detailed derivations make an order of magnitude estimate of the electron-ion collision time at the points of reflection.   [4]

2. Particles of charge q move in a region of space where there is a uniform magnetic field, $(0,0, B_z)$ and uniform electric field, $(0, E_y, 0)$. Show that the guiding centre of the particles drifts with a uniform velocity, and derive an expression for the magnitude and direction of this drift. [6]

Use the above result to derive the drift velocity of the particles when the electric field is replaced by a gravitational field, commenting on any differences arising from the sign of the charge on the particle. [4]

The effective additional force $\mathbf{F_B}$ on a charged particle owing to a gradient in the magnetic field is given by

$$\mathbf{F_B} = \frac{mv_\perp^2}{2B} \nabla \mathbf{B} \,,$$

where $v_\perp$ is the velocity of the particle perpendicular to the magnetic field. Protons are situated in the equatorial plane within a plasma that surrounds a planet of mass M. The planet has a magnetic field that can be approximated as a dipole, with the magnetic field strength varying as $B = \mu/r^3$, where r is the distance to the centre of the planet. Show that for the magnitude of the grad-B drifts and gravitational drifts to be equal, $r = AM/v_\perp^2$, and obtain an expression for A. [10]

Hence show that for protons of even modest energies (of order an eV) surrounding the earth the grad-B drift dominates over the gravitational drift. [5]

[The radius of the earth may be taken to be $6370\,\mathrm{km}$, and its mass $5.9{\times}10^{24}\,\mathrm{kg}$.]

3. Show that when a high power laser with intensity I and wavelength λ ablates material from a target the ablation pressure produced is given by

$$P_A = C(I/\lambda)^{2/3} \,,$$

where C is a constant. Obtain an estimate for C, and evaluate the ablation pressure in Mbar for a laser with wavelength 0.35 μm operating at an intensity of $10^{15}\,\mathrm{W\,cm^{-2}}$. (You may assume that the ablation velocity of the plasma at the critical density surface is Mach 1.)   [8]

The Lawson criterion for energy gain in a Deuterium-Tritium (D-T) fusion reaction states that the product of the number density of ions, n, and the time for which they are confined, τ, is given by $n\tau > 10^{20}\,\mathrm{m^{-3}\,s}$. For fusion to occur the D-T fuel must be heated to temperatures of order 10keV. Show that for inertial confinement fusion, where a sphere of D-T of radius r and density ρ must be heated, the Lawson criterion may be written as $\rho r > A$, and evaluate a rough value of A in units of $\mathrm{g\,cm^{-2}}$.   [6]

The National Ignition Facility (NIF) in California is a high power laser that operates at a wavelength of 0.35 μm, and can deliver up to $2\,\mathrm{MJ}$ of laser energy in pulses of durations of order $10\,\mathrm{nsec}$. Demonstrate that this laser is incapable of producing fusion simply by heating a solid D-T target of density $0.2\,\mathrm{g\,cm^{-3}}$, but could potentially produce fusion in a target of in excess of 100 times solid density. Estimate the Fermi pressure of the fuel at this density and compare it with the ablation pressure, assuming NIF is limited to operating at an absorbed intensity of order $10^{14}\,\mathrm{W\,cm^{-2}}$.   [6]

Explain how the high pressures for fusion might be achieved, and comment on what physical processes hinder the attainment of this goal.   [5]

Section S18   ADVANCED QUANTUM MECHANICS

1. In natural units ($\hbar = c = 1$) the Klein-Gordon equation for a particle of mass m and charge e moving in an electromagnetic field with 4-vector potential $A_\mu = (V, \mathbf{A})$ is

$$(\partial_\mu + \mathrm{i}eA_\mu)(\partial^\mu + \mathrm{i}eA^\mu)\psi + m^2\psi = 0.$$

Show that the current

$$j_\mu = \mathrm{i}(\psi\partial_\mu\psi^* - \psi^*\partial_\mu\psi) + 2eA_\mu\psi^*\psi$$

satisfies $\partial^\mu j_\mu = 0$. [6]

Show that j_μ is left unchanged by the transformations

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \partial_\mu\phi \\ \psi &\rightarrow \mathrm{e}^{\mathrm{i}e\phi}\,\psi \\ \psi^* &\rightarrow \mathrm{e}^{-\mathrm{i}e\phi}\,\psi^* \end{aligned}$$

where ϕ is an arbitrary function. [4]

For $V = 0$ and $\mathbf{A} = (-y, x, 0)B/2$, where B is a constant, compute the electric and magnetic field strengths. In this case show that the wave function

$$\psi = C\mathrm{e}^{-(x^2+y^2)eB/4 - \mathrm{i}Et},$$

where C is a normalization constant, is a solution of the Klein-Gordon equation and find the possible energies E of the particle and the current j_μ for this solution. [10]

What is the motion described by this wave function? Find the energy shift induced by B in the non-relativistic limit and give a semi-classical explanation for it. [5]

2. Consider scattering of a particle of mass m and wavenumber k incident from the negative z direction on a potential $V(r)$ which falls off rapidly at large r. Assuming that the wavefunction $\psi(r, \theta, \phi)$ takes the form

$$\psi(r, \theta, \phi) = e^{ikz} + \frac{e^{ikr}}{r} f_k(\theta, \phi)$$

at very large r show that the differential cross-section is given by

$$\frac{\partial \sigma}{\partial \Omega} = |f_k(\theta, \phi)|^2 .$$

[5]

The wavefunction for a plane wave of wavenumber k propagating in the z direction can be written in the partial wave expansion as

$$e^{ikz} = \sum_{\ell=0}^{\infty} (2\ell + 1) \frac{1}{2ikr} \left(e^{ikr} - (-1)^{\ell} e^{-ikr} \right) P_{\ell}(\cos \theta)$$

for $r \gg k^{-1}$. Explain what is meant by the *phase shift* δ_{ℓ} and show that the total cross section σ is given by

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_{\ell} .$$

Hence prove the Optical Theorem which states that

$$\sigma = \frac{4\pi}{k} \, \mathrm{Im} f_k(\theta = 0) .$$

[8]

In the first Born approximation

$$f_k(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3 \mathbf{r}' \, V(\mathbf{r}') e^{i(\mathbf{k}_i - \mathbf{k}_f) \cdot \mathbf{r}'}$$

where $\mathbf{k}_i$ and $\mathbf{k}_f$ are the wavevectors of the incoming and scattered particle respectively. Compute $f_k(\theta, \phi)$ and the total cross section σ for the Yukawa potential

$$V(\mathbf{r}) = V_0 \frac{e^{-\alpha r}}{r} .$$

[8]

Find $\mathrm{Im} f_k(\theta = 0)$ to the second Born approximation for the Yukawa potential. [4]

$$\left[\int_{-1}^{1} \mathrm{d}x \, P_{\ell}(x) \, P_{\ell'}(x) = \frac{2\delta_{\ell\ell'}}{2\ell + 1}, \quad P_{\ell}(0) = 1 \right]$$

3. In natural units ($\hbar = c = 1$) the Dirac Hamiltonian for a free particle of mass m is given by

$$H = \boldsymbol{\alpha}.\mathbf{p} + m\beta.$$

Show that the eigenvalues, E, of H satisfy $E^2 = \mathbf{p}^2 + m^2$ provided

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}I, \quad \alpha_i\beta + \beta\alpha_i = 0, \quad \beta^2 = I. \tag{1}$$

[5]

Assuming that the matrices $\boldsymbol{\alpha}$ and β satisfy the constraints (1) show that $\boldsymbol{\alpha}' = U^\dagger\boldsymbol{\alpha}U$ and $\beta' = U^\dagger\beta U$ also satisfy them provided U is unitary.

Find the matrix U which relates the Dirac basis

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

to the Weyl basis

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

If ψ_{D} is an eigenfunction of H in the Dirac basis what is the corresponding eigenfunction ψ_{W} in the Weyl basis?

[8]

Find the eigenvalues $\lambda_\pm$ and the corresponding normalized eigenspinors $u_\pm$ of $\boldsymbol{\sigma}.\mathbf{p}$. [5]

Show that in the Weyl basis the eigenspinors of H may be written

$$N_\pm \begin{pmatrix} u_\pm \\ \frac{E\mp p}{m}u_\pm \end{pmatrix}, \quad N_\mp \begin{pmatrix} u_\pm \\ -\frac{(E\pm p)}{m}u_\pm \end{pmatrix},$$

where $E = +\sqrt{\mathbf{p}^2 + m^2}$. Find the normalization constants $N_\pm$ and show that these spinors form an orthonormal basis.

[7]

$$\left[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

Section S19   PARTICLE ACCELERATOR SCIENCE

The following formula may be useful:

Critical photon energy of synchroton radiation:

$$\varepsilon_c = \frac{3}{2}\frac{\hbar c}{\rho}\gamma^3,$$

where γ is the relativistic factor of the electron beam and ρ is the dipole radius of curvature.

---

1. Consider a FODO cell made, in the order described, of a quadrupole with focal length f, a drift of length L, a quadrupole with focal length $-f$, and a drift of length L. The quads can be treated in the thin lens approximation.

(a) Calculate the transfer matrix of the cell.   [8]

(b) Calculate the phase advance of the cell. What is the phase advance of the full cell if we start from a different point in the cell? [Hint: use the Twiss representation for the one turn map.]   [7]

(c) Calculate the beta functions at the beginning of the cell. Where are the maximum and minima of the beta functions located? How does the ratio between maximum and minimum beta vary with the phase advance?   [6]

Explain the difference between geometric aperture and dynamic aperture. What determines each of them? Can the dynamic aperture be equal to the geometric aperture?   [4]

2. The Diamond light source consists of three accelerators: a 100 MeV Linac, a 3 GeV Booster and a 3 GeV storage ring.

(a) The emittance of the electron beam injected into the storage ring from the booster is 140 nm, while the equilibrium emittance in the storage ring is 2.7 nm. Explain what happens to the emittance of the injected electron beam and why. If we inject a proton beam with 140 nm emittance what happens to the emittance? [6]

(b) The Diamond dipole radius is 7.1 m. What is the critical energy for a 3 GeV electron beam and what is it if we have a proton beam? What is the beam energy that maximises the photon flux for a user who is interested in photons with 10 eV energy? [7]

(c) Calculate the energy loss per turn for a 3 GeV beam at Diamond. What is the total RF power needed if we want to store a beam of 300 mA? Assume the RF trips. If the maximum dispersion is 0.3 m and the aperture is 10 cm how many turns it will take for the beam to hit the vacuum chamber? [7]

(d) Calculate the longitudinal damping time for 3 GeV electrons and protons in the Diamond storage ring. [5]

3.

(a) Because it is commonly available, relatively inexpensive and an excellent conductor, copper is used extensively in 'warm' waveguides and cavities. The conductivity of copper is $\sigma_c = 5.96 \times 10^7 \, \Omega^{-1} \, \mathrm{m}^{-1}$.
For a common S-band frequency, $f_{\mathrm{RF}} = 2.856$ GHz. What is the skin depth and surface resistivity of copper at this frequency? [4]

(b) A new 50 GeV (kinetic energy) proton synchrotron, the PS2 accelerator, is being designed at CERN to replace the PS accelerator. The new accelerator will sit in a new ring tunnel which will have a mean radius of 215 m and will receive an injected beam at 4 GeV (kinetic energy) from a new accelerator, the Superconducting Proton Linac (SPL).
Given that the mass of the proton is 0.9383 GeV/c^2:

  • What is the revolution frequency at 4 GeV, 20 GeV and at 50 GeV?
  • If the harmonic number is 32, what is the RF frequency at 4 GeV and at 50 GeV for a synchronous phase $\varphi_s = 0°$?
  • For a representative 10 MHz PS2 cavity, what is the transit-time factor for protons of 4 GeV (kinetic energy)? [9]

(c) The beam monitors described in the lectures can be used to measure the optical parameters of the beam. Provide a description of how the following optical parameters of the beam can be measured:

  • Measurement of the tunes.
  • Measurement of the beta function.
  • Measurement of the dispersion.
  • Measurement of the chromaticity. [12]

Section S25   PHYSICS OF CLIMATE CHANGE

1. Explain what is meant by the *spectrally integrated brightness temperature* of a planet, T_e. Show that T_e for the Earth is given by the expression

$$\sigma T_e^4 = \frac{S_0(1-\alpha)}{4} \ ,$$

where S_0 is the solar constant and α is the planetary albedo, explaining the origin of the factor of 4 on the right-hand side. Calculate T_e for the Earth assuming $\alpha = 0.3$.   [5]

Show, by considering the energy budget of a horizontal slab of atmosphere with spectrally-integrated emissivity ϵ_1, where $\epsilon_1 \ll 1$, that you would expect to find a region of constant temperatures in the upper atmosphere of a planet in which carbon dioxide is the only radiatively active gas and there is no significant vertical transport of heat due to dynamical activity. Find an expression for the temperature T_{strat} of this "stratosphere" region in terms of S_0 and α and evaluate it for $\alpha = 0.3$.   [5]

Now assume this stratospheric "slab" also contains ozone, which absorbs short-wave radiation with emissivity $\epsilon_2 \ll 1$. You may neglect the impact of ozone on infra-red radiation and assume the ozone layer is entirely opaque in the wavenumbers it absorbs, and is otherwise transparent. Show that the resulting shortwave heating of the slab, in $W\,m^{-2}$, is given by $\epsilon_2 S_0/4$. If temperatures in the Earth's stratosphere are found to be $80\,K$ warmer than the value for T_{strat} evaluated in the previous section, due solely to this ozone-induced heating, and α remains 0.3, evaluate the ratio ϵ_2/ϵ_1.   [6]

Assuming absorption by ozone and absorption and emission by carbon dioxide are the only sources of heating and cooling of the stratosphere, calculate to the nearest K the change in T_e and T_{strat} caused by (a) a change in planetary albedo from 0.3 to 0.31 and (b) a doubling of carbon dioxide concentrations, with α held fixed at 0.3.   [9]

2. An instantaneous doubling of atmospheric carbon dioxide concentrations reduces net upward infra-red radiation at the tropopause by about $4\,\mathrm{W\,m^{-2}}$ and at the surface by about $1\,\mathrm{W\,m^{-2}}$. Explain qualitatively why these two numbers are different. If there is no change in surface or tropospheric temperature and no change in sensible heat fluxes at the surface following the CO_2-doubling, calculate the resulting change in global-mean precipitation in millimetres per day assuming all precipitation is in the form of liquid water. [You may assume the latent heat of vaporisation of water is constant at $2500\,\mathrm{kJ\,kg^{-1}}$]. [4]

A simple model of the rate of change of fractional rainforest cover C in Amazonia is given by

$$
\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}t} &= \gamma(P - P_{\mathrm{c}}) - \beta C^3 \quad \text{for} \quad 0 < C \le 1 \,, \\
&= 0 \quad \text{for} \quad C = 0 \,,
\end{aligned}
$$

where P is the precipitation and γ, β and P_{c} are positive constants. What is the physical interpretation of P_{c}? Precipitation is represented by $P = P_0 + \alpha C$, where P_0 is the background precipitation supplied by the large-scale atmospheric circulation and αC represents the enhancement of precipitation through local moisture recirculation. Assuming that $\beta > \gamma(P_0 + \alpha - P_{\mathrm{c}})$ and that $P_0 < P_{\mathrm{c}}$, sketch $\mathrm{d}C/\mathrm{d}t$ as a function of C. Hence, or otherwise, show that the model supports three equilibrium values of C, two of which are stable, provided P_0 exceeds a certain critical value, P_{d}, and derive an expression for P_{d}. [11]

Assuming Amazonia is initially forested and in equilibrium, describe with the aid of a sketch what happens to C if P_0 is gradually and steadily reduced to a value less than P_{d}. Discuss, with the aid of the same sketch, what happens to C if P_0 is subsequently increased to a value greater than P_{c} (a) if $\mathrm{d}C/\mathrm{d}t$ conforms exactly to the above model or (b) if active conservation efforts mean that C is prevented from falling below a minimum value $C_{\min} \ll 1$. [6]

If complete deforestation of Amazonia releases into the atmosphere 100 billion tonnes ($10^{14}\,\mathrm{kg}$) of carbon, half of which is immediately taken up by the ocean and land biosphere, calculate the resulting percentage change in atmospheric carbon dioxide concentration. [Assume that the initial atmospheric mass-mixing ratio of carbon dioxide is $5.7 \times 10^{-4}\,\mathrm{kg/kg}$.] [4]

3. A simple model of the transient global temperature response to an externally imposed radiative forcing perturbation $F(t)$ is given by

$$C\frac{\mathrm{d}T(t)}{\mathrm{d}t} + \lambda T(t) = F(t) \, ,$$

where $T(t)$ is the global temperature anomaly with respect to the value to which it equilibrates if $F = 0$. Explain the physical significance of the constants C and λ. [4]

A unit mass of a greenhouse gas is suddenly injected into the atmosphere at $t = 0$, mixes rapidly though the atmosphere and is subsequently destroyed by sink processes that cause its concentration to decay exponentially back to its equilibrium value with rate constant k_1, inducing a radiative forcing perturbation $F(t) = F_0 e^{-k_1 t}$. Show that the temperature response under the above simple model is given by

$$T = T_0 \, \frac{e^{-k_1 t} - e^{-k_0 t}}{k_0 - k_1}$$

and find expressions for the constants T_0 and k_0, where k_0^{-1} is known as the *feedback response time* of the climate system. What happens when $k_0 = k_1$? [8]

Emission trading systems use the 100-year Global Warming Potential, or GWP-100, to compute nominally equivalent masses of different greenhouse gases. GWP-100 is defined as the integrated radiative forcing over the next 100 years resulting from the sudden injection of a unit mass of a greenhouse gas. Following a pulse injection, concentrations of both HCFC-22 and CFC-12 decay exponentially with (you may assume) approximate lifetimes of 10 and 100 years respectively. Given that the 100-year Global Warming Potential of CFC-12 is 6 times that of HCFC-22, calculate the ratio of the radiative forcing induced by a unit mass injection of HCFC-22 to the radiative forcing induced by a unit mass injection of CFC-12, evaluated in both cases immediately after the injection occurs. [4]

Assuming the feedback response time of the climate system k_0^{-1} is 20 years, calculate the ratio of the warming induced by a unit mass injection of HCFC-22 to the warming induced by a unit mass injection of CFC-12 at (a) 20 years and (b) 100 years after the injection occurs, and comment on your result. [9]

Section S26   STARS AND GALAXIES

1. Derive the equation of hydrostatic equilibrium within a star, at a radius r enclosing a mass M_r. Use this to estimate the central temperature and pressure of the Sun, explaining any assumptions you make. [7]

Describe the process by which helium is created in the solar interior. If two protons must come within a nuclear radius of each other to fuse, estimate the average kinetic energy per proton. What temperature would this correspond to? Compare your answer with your previous estimate of the central temperature of the Sun, and explain any discrepancy. [7]

What is meant by the term *Main Sequence Star*? For massive stars on the main sequence, using simple scaling relations and stating any assumptions you make, derive the relation between luminosity (L) and stellar mass (M), $L \propto M^k$ (where k is a constant to be determined). You may assume that within massive stars radiative transfer dominates the energy transport, and the temperature gradient is given by

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{-3\,\kappa\,\rho\,L_r}{16\pi\,a\,c\,r^2\,T^3}\,,$$

where κ is the opacity, ρ the density, L_r is the luminosity within a radius r and a is the radiation constant ($a = 4\sigma/c$ in terms of σ, the Stefan-Boltzmann constant). [5]

Explain what is meant by the *main sequence turn-off* in a Herzsprung-Russell diagram of a star cluster (where all the stars have the same age), and describe how this evolves with the age of a cluster.

The Pleiades (or Seven Sisters) is an "open star cluster" with a parallax of 0.0075 arcseconds (measured by Hipparcos). The brightest main sequence star has an apparent magnitude of 4.3 in the V-band and a spectral type of B6V. Estimate the age of this cluster, stating any assumptions you make. [6]

[You may take the absolute bolometric magnitude of the Sun to be $M_{bol} = 4.6$, and the bolometric correction for the V-band for a B6V star to be -1.2 mag.]

2. What is meant by a *white dwarf* star? Explain the terms *degenerate* and *relativistic*. For a fully-relativistic, degenerate white dwarf interior derive the equation of state (that is, the relation between pressure, P, and density, ρ).

Explain, with reasons, the temperature profile with radius of a white dwarf star. [9]

For a non-relativistic white dwarf,

$$P \propto \rho^{\frac{5}{3}} \ .$$

How does the radius of a non-relativistic white dwarf scale with mass? Discuss the behaviour of the mass–radius relation in the fully-relativistic case. [7]

The star Sirius, at a distance from Earth of 2.6 parsec, has a white dwarf companion (Sirius B) whose mass is $0.98\,M_\odot$. Given that Sirius B has an effective surface temperature of 25,000 K and an apparent magnitude in V of 8.3 mag, estimate its radius. [3]

[You may take the absolute bolometric magnitude of the Sun to be $M_{bol} = 4.6$ and the bolometric correction for Sirius B to be -2.5 mag in the V-band.]

The luminosity, L_{WD}, of a white dwarf is related to its core temperature, T_c, by

$$L_{WD} = 0.09 \times \frac{M}{M_\odot} T_c^{\frac{7}{2}} \ \text{W} \ .$$

Calculate the time for the effective surface temperature of Sirius B to cool to the surface temperature of the Sun. [6]

3. Describe with the aid of a diagram the Hubble classification scheme for galaxies. Indicate for each main type the level of ongoing star formation. [4]

Draw a sketch of what the Milky Way galaxy would look like viewed from a distance, labelling the main structural features and giving approximate sizes of these components and the age ranges and metallicities for stars in these regions. [5]

Derive the collisionless Boltzmann equation for the distribution function, $f(r, \theta, z, v_r, v_\theta, v_z)$, in cylindrical coordinates r, θ, z, where v_r, v_θ, v_z are the velocity components in those directions.

Hence, show that for an axisymmetric galaxy in steady state:

$$\frac{\partial(n \langle v_r\, v_z\rangle)}{\partial r} + \frac{\partial(n \langle v_z^2\rangle)}{\partial z} + \frac{n \langle v_r\, v_z\rangle}{r} = -n\frac{\partial\Phi}{\partial z} \ ,$$

where Φ is the gravitational potential, n is the number density of stars, and $\langle\ \rangle$ denotes an average. [10]

In our Galaxy, the stellar velocity components v_r and v_z are independent (for small motions). Show that $\langle v_r\, v_z\rangle = 0$ in steady state, and hence deduce that in the Galactic disk:

$$\frac{\partial}{\partial z}\left(\frac{1}{n}\frac{\partial(n\langle v_z^2\rangle)}{\partial z}\right) = -4\pi\, G\, \rho \ .$$

How could this be applied to measuring the mass density in the solar neighbourhood? [6]